Notes on Categories

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18th August 2004

Contents

Part I

Basic Definitions

1 Categories

1.1 Definition and Examples

In this section we provide the definition of category and many examples of categories which may already be familiar to the reader.

1.1 Definition. A *category* $\mathbf{C}$ consists of

1. A class $\operatorname{obj}\mathbf{C}$, whose elements are called *objects*.
2. A set $\hom_{\mathbf{C}}(A, B)$ for every pair of objects A, B, whose elements are called *morphisms*, or *maps* from A to B. We can call them also $\mathbf{C}$-maps.
3. For every triad of objects A, B, C, a function (called *composition*)
$$\hom_{\mathbf{C}}(A, B) \times \hom_{\mathbf{C}}(B, C) \to \hom_{\mathbf{C}}(A, C) \tag{1.1}$$
whose value at (f, g) will be denoted by $g \circ f$.
4. For every object A, a distinguished element $1_A^{\mathbf{C}} \in \hom_{\mathbf{C}}(A, A)$, called *identity* on A.

They have to satisfy the following conditions:

(a) For every pair of objects A, B and for every $f \in \hom_{\mathbf{C}}(A, B)$, we must have $f \circ 1_A^{\mathbf{C}} = 1_B^{\mathbf{C}} \circ f = f$.

(b) For every $A, B, C, D \in \operatorname{obj}\mathbf{D}$ and $f \in \hom_{\mathbf{C}}(A, B)$, $g \in \hom_{\mathbf{C}}(B, C)$ and $h \in \hom_{\mathbf{C}}(C, D)$ we have that $h \circ (g \circ f) = (h \circ g) \circ f$.

An identity $1_A^{\mathbf{C}}$ is usually denoted by 1_A if the category **C** is clear from the context. If A and B are objects in the category **C**, then $f\colon A \to B$ will mean that f is an element of $\hom_{\mathbf{C}}(A, B)$. In that case, we say that A is the *domain* of f and that B is the *codomain* of f, and we write $A = \operatorname{dom} f$, $B = \operatorname{cod} f$.

An element of obj **C** will be called a **C**-object, similarly a morphism in **C** will be called a **C**-morphism.

We note that we always consider different hom sets to be disjoint. That is, if f, g are morphisms in a category **C** such that $f = g$, with $f \in \hom_{\mathbf{C}}(A, B)$, $g \in \hom_{\mathbf{C}}(C, D)$, then we must have that $A = C$, $B = D$.

Our first example is the prototype of all categories.

Example **1.1**

The category **Set** of sets, such that obj **Set** is the class of all sets and $\hom_{\mathbf{Set}}(A, B)$ is the set of all maps from A to B. Composition is the usual composition of maps, and for a set X, the identity 1_X is the identity map. Then clearly the conditions (a) and (b) of the definition of categories are satisfied.

In the following examples of categories, the objects are sets with some extra structure, and morphisms are maps which preserve that structure. Those categories are called *concrete*, in them, the compositions are given by the usual composition of maps, and the identities are the identity maps. Since the usual composition is associative and the identity map is a neutral element under composition, in order to prove that we have a category one only has to check that the composition of two morphisms is a morphism, and that the identity map is a morphism.

Examples from Algebra

Examples **1.2**

1. The category **Grp** of all groups, where we have that obj **Grp** is the class of all groups and $\hom_{\mathbf{Grp}}(A, B)$ is the set of group homomorphisms from A to B.
2. Fixing a group G, we have the category **GSet**, where obj **GSet** is the class of (left) G-sets and $\hom_{\mathbf{GSet}}(X, Y)$ is the set of equivariant maps.
3. The category **Ab** of all abelian groups, such that obj **Ab** is the class of all abelian groups and $\hom_{\mathbf{Ab}}(A, B)$ is the set of group homomorphisms from A to B.
4. The category **Rng** of rings, such that obj **Rng** is the class of all rings, $\hom_{\mathbf{Rng}}(A, B)$ is the set of ring homomorphisms from A to B.

5. The category **Ring** of rings with unit, such that obj **Ring** is the class of all rings with unit and $\hom_{\mathbf{Ring}}(A, B)$ is the set of all ring homomorphisms from A to B such that $f(1) = 1$.
6. The category R-**mod**, where R is a ring, such that obj R-**mod** is the class of all left modules over R and $\hom_{R\text{-}\mathbf{mod}}(A, B)$ is the set of R-linear morphisms from A to B. We similarly have a category **mod**-R of right R-modules. Note that if R is in fact a field, then R-**mod** is the category of vector spaces over R, and the morphisms in this particular case are linear transformations.
7. The category R-**alg** of R-algebras, such that obj R-**alg** is the class of all algebras over R and $\hom_{R\text{-}\mathbf{alg}}(A, B)$ is the set of morphisms of R-algebras from A to B.

Examples from Topology

Examples **1.3**

1. The category **Top** of topological spaces, such that obj **Top** is the class of all topological spaces and $\hom_{\mathbf{Top}}(X, Y)$ is the set of continuous maps from X to Y.
2. The category $\mathbf{Top}_*$ of pointed topological spaces, such that obj $\mathbf{Top}_*$ is the class of all pairs (X, x_0), where X is a topological space and x_0 is a point in X. We put $\hom_{\mathbf{Top}_*}\big((X, x_0), (Y, y_0)\big)$ as the set of continuous maps from X to Y such that $f(x_0) = y_0$.
3. The category **Haus** of Hausdorff topological spaces, such that obj **Haus** is the class of all Hausdorff topological spaces and $\hom_{\mathbf{Haus}}(X, Y)$ is the set of all continuous maps from X to Y.
4. The category **Metric** of metric spaces, such that obj **Metric** is the class of all metric spaces and $\hom_{\mathbf{Metric}}(X, Y)$ is the set of all continuous maps (satisfying an ε-δ definition) from X to Y.

Examples from Combinatorics

Examples **1.4**

1. The category **Poset**, with class of objects the class of partially ordered sets and $\hom_{\mathbf{Poset}}(P, Q)$ is the set of monotone maps from P to Q.
2. The category **Graph** where obj **Graph** is the class of all graphs, and for $G_1, G_2 \in$ obj **Graph**, we have that $\hom_{\mathbf{Graph}}(G_1, G_2)$ is the set of functions from G_1 to G_2 that preserve adjacency.
3. The category **DirGraph** where obj **DirGraph** is the class of all directed graphs, and for $G_1, G_2 \in$ obj **Graph**, we have that $\hom_{\mathbf{Graph}}(G_1, G_2)$ is the set of functions f such that whenever $a \to b$ is and edge in G_1, then $f(a) \to f(b)$ is an edge in G_2.
4. The category **SimplComplex**, with class of objects the class of abstract simplicial complexes and $\hom_{\mathbf{SimplComplex}}(K, L)$ is the set of simplicial maps from K to L.

At this point, we should now provide examples of categories which are not concrete. In the following examples, either the objects are not sets with structure or the hom sets are not composed by maps between sets.

Examples **1.5**

1. We define a category **mod** where obj **mod** consists of pairs (R, M) where R is a ring and M is a left R-module. A morphism $(R, M) \to (S, N)$ in **mod** is a pair of maps (ϕ, f), where $\phi\colon R \to S$ is a morphism of rings and $f\colon M \to N$ is an additive map such that $f(rm) = \phi(r)f(m)$ for all $r \in R$, $m \in M$. We define composition as $(\phi', f') \circ (\phi, f) = (\phi' \circ \phi, f \circ f')$, which can be checked it is well-defined, and $1_{(R,M)} = (1_R, 1_M)$.
2. Consider the category **Toph**, where the class of objects is the class of topological spaces and the set $\hom_{\mathbf{Toph}}(X, Y)$ is the set of homotopy classes of continuous maps from X to Y. The composition of homotopy classes is defined to be the class of the composition of arbitrary representatives. This is well defined, since homotopy of maps is an equivalence relation preserved by composition. See [ML98, page 52]. We set $1_X^{\mathbf{Toph}}$ as the homotopy class of the identity map $X \to X$. The properties of homotopy let us check the conditions for a category.
3. Let **Rel** be a category such that obj **Rel** = obj **Set**, and for sets X, Y, we have that $\hom_{\mathbf{Rel}}(X, Y)$ is the set of relations from X to Y. If $R\colon X \to Y$ and $S\colon Y \to Z$, then $S \circ R\colon X \to Z$ is given by

$$S \circ R = \{ (x, z) \in X \times Z \mid \text{there is } y \in Y \text{ with } (x, y) \in R, (y, z) \in S \}. \tag{1.2}$$

   We put $1_X = \{ (x, x) \mid x \in X \}$. It is straightforward to check that we have just defined a category.

We will have plenty of more examples in the next section.

✎ Exercises 1.1

1. Let **C** be a category, and $A \in \operatorname{obj} \mathbf{C}$. If $h \in \hom_{\mathbf{C}}(A, A)$ has the property that $h \circ f = f$ for every $f \in \hom_{\mathbf{C}}(A, A)$, then $h = 1_A^{\mathbf{C}}$.

1.2 Small Categories

1.2 Definition. A category **C** is *small* if obj **C** is a set.

All the categories defined in the previous section are not small, hence they are called *large*.

Consider the following examples of small categories:

Examples **1.6**

1. The empty category $\mathbf{0}$, with no objects (hence, no morphisms).
2. The category $\mathbf{1}$ such that $\operatorname{obj} \mathbf{1} = \{*\}$ (a set with one element), and $\hom_{\mathbf{1}}(*, *) = \{1_*\}$. The composition is then uniquely defined. The composition is uniquely defined if it is going to satisfy (a) from Definition 1.1.
3. The category $\mathbf{2}$ with $\operatorname{obj} \mathbf{2} = \{*, *'\}$ and one non-identity map $* \to *'$. Again, the composition can only be defined in one way in order form $\mathbf{2}$ to be a category.
4. Similarly, we can define a small category by means of the diagrams $\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$, $\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$, or $\bullet \rightrightarrows \bullet$. In every case, each dot represents a different object in the category being defined, and the arrows are the only non-trivial morphisms.
5. Let G be a group. We define a category $\mathbf{G}$ by $\operatorname{obj} \mathbf{G} = \{*\}$, $\hom_{\mathbf{G}}(*, *) = G$, composition equal to the group multiplication and $1_* =$ identity element of G. We say that $\mathbf{G}$ is the category *associated* to the group G.
6. Let P be a preordered set, that is, a set with a reflexive and transitive relation denoted by $\leq$. We define a category $\mathbf{P}$ by putting $\operatorname{obj} \mathbf{P} = P$, and

$$\hom_{\mathbf{P}}(x, y) = \begin{cases} (x \leq y) & \text{if } x \leq y \\ \emptyset & \text{if } x \not\leq y \end{cases}$$

   The composition will be given by $(y \leq z) \circ (x \leq y) = (x \leq z)$, and $1_x = (x \leq x)$. We say that $\mathbf{P}$ is the category *associated* to the preordered set P.

   In particular, if $n \geq 0$, we will use $[n]$ to denote the category associated to the subposet $\{0, 1, \ldots, n\}$ of $\mathbb{N} \cup \{0\}$ with the usual order relation.
7. Let X be a topological space, then we can define a category $\mathbf{X}$ with $\operatorname{obj} \mathbf{X} = X$, and $\hom_{\mathbf{X}}(x, y)$ the set of homotopy classes of continuous maps (paths) $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x$, $\gamma_1 = y$. The composition is defined in representatives as the usual concatenation of paths and the identity 1_x is the class of the constant path with value x.

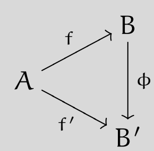
1.3 Definition. A *discrete category* is a category in which all the morphisms are identities.

There can be discrete large categories, but usually this definition will be of use to us when we consider any set X as a small discrete category.

1.3 Constructions

We show now some ways of forming new categories from old ones.

1.4 Definition. Let **C** be a category and $A \in \operatorname{obj} \mathbf{C}$. We define the comma category $A \downarrow \mathbf{C}$, where $\operatorname{obj} A \downarrow \mathbf{C}$ is the class of all morphisms in **C** with domain A. If $f\colon A \to B, f'\colon A \to B' \in \operatorname{obj} A \downarrow \mathbf{C}$, then $\hom_{A \downarrow \mathbf{C}}(f, f')$ is the set of **C**-morphisms $\phi\colon B \to B'$ such that $\phi \circ f = f'$. Using diagrams, this last statement is equivalent to saying that the following diagram:



(1.3)

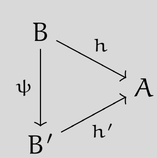
commutes.

It is entirely possible that the same **C**-morphism ϕ makes commute two different diagrams of the form 1.3. However, as noted after Definition 1.1, we must consider them as two different $A \downarrow \mathbf{C}$-morphisms.

Example **1.7**

As an example of a comma category, let $*$ be a topological space with just one point. Then $* \downarrow \mathbf{Top}$ consists of maps of the form $* \to X$, which can be identified with X together with a choice of a basepoint (the image of $*$). And the morphisms in $* \downarrow \mathbf{Top}$ are continuous maps that preserve the basepoint. Hence, in some sense, $* \downarrow \mathbf{Top}$ can be identified with the category $\mathbf{Top}_*$.

1.5 Definition. Similarly, if **C** be a category and $A \in \operatorname{obj} \mathbf{C}$, we define a comma category $\mathbf{C} \downarrow A$, where $\operatorname{obj} \mathbf{C} \downarrow A$ is the class of all morphisms in **C** with codomain A. If $h\colon B \to A, h'\colon B' \to A \in \operatorname{obj} \mathbf{C} \downarrow A$, then $\hom_{\mathbf{C} \downarrow A}(h, h')$ is the set of **C**-morphisms $\psi\colon B \to B'$ that make the following diagram commute:



(1.4)

1.6 Definition. Let **C** be a category. The *opposite category* $\mathbf{C}^{op}$ has as objects the same class of objects as **C**, the hom sets are defined by $\hom_{\mathbf{C}^{op}}(A, B) = \hom_{\mathbf{C}}(B, A)$, the composition $g \circ f$ in $\mathbf{C}^{op}$ is defined to be equal to the composition $f \circ g$ in **C**, and the identities are the same as in **C**.

Example 1.8

It is straightforward that $\left(\begin{array}{cc} & \bullet \\ & \downarrow \\ \bullet \longrightarrow & \bullet \end{array} \right)^{op} = \begin{array}{cc} \bullet \longrightarrow & \bullet \\ \downarrow & \\ \bullet & \end{array}$.

1.7 Definition. Let **C** and **D** be categories. The *product category* $\mathbf{C} \times \mathbf{D}$ has as objects the pairs (A, A') with $A \in \operatorname{obj} \mathbf{C}$, $A' \in \operatorname{obj} \mathbf{D}$, the morphisms from (A, A') to (B, B') are pairs of morphisms (f, f') with $f \colon A \to B$ in **C** and $f' \colon A' \to B'$, composition is defined componentwise, and $1_{(A,A')} = (1_A, 1_{A'})$.

Example 1.9

For example, if G_1, G_2 are groups, the category associated to the direct product of groups $G_1 \times G_2$ is the product category $\mathbf{G_1} \times \mathbf{G_2}$

More generally, for any set of n-categories $\mathbf{C}_1, \dots, \mathbf{C}_n$, we can define a category $\mathbf{C}_1 \times \cdots \times \mathbf{C}_n$. We will denote the product $\mathbf{C} \times \cdots \times \mathbf{C}$ with n factors as $\mathbf{C}^{\times n}$.

1.8 Definition. Let **C** be a category such that there is an equivalence relation $\simeq$ defined on each $\hom_{\mathbf{C}}(A, B)$ for any pair of objects A, B in **C**, with the property that if $f, f' \in \hom_{\mathbf{C}}(A, B)$ with $f \simeq f'$ and $g \colon A' \to A$, $h \colon B \to B'$ are maps, then $hfg \simeq hf'g$. (In this case $\simeq$ is called a *congruence* in **C**). Then there is a *quotient category* $\mathbf{C}/\simeq$ with the objects the same objects as in **C**, the hom sets are the equivalence classes under the relation $\simeq$, the composition is defined by composition of representatives, and for any $A \in \operatorname{obj}(\mathbf{C}/\simeq) = \operatorname{obj} \mathbf{C}$, the identity on A is the class of 1_A in $\hom_{\mathbf{C}}(A, A)$.

Example 1.5.2 is an instance of this definition.

1.4 Subcategories

1.9 Definition. Let $\mathbf{C}$, $\mathbf{C}'$ be categories. We say that $\mathbf{C}'$ is a *subcategory* of $\mathbf{C}$ if $\operatorname{obj}\mathbf{C}'$ is a subclass of $\operatorname{obj}\mathbf{C}$, $\hom_{\mathbf{C}'}(A, B)$ is a subset of $\hom_{\mathbf{C}}(A, B)$ for all A, B in $\operatorname{obj}\mathbf{C}'$, the compositions in $\mathbf{C}'$ are defined and are restrictions of the corresponding compositions in $\mathbf{C}$, and the identity morphisms in $\mathbf{C}'$ are the ones that are so in $\mathbf{C}$.
If $\mathbf{C}'$ is a subcategory of $\mathbf{C}$ and we have that $\hom_{\mathbf{C}'}(A, B) = \hom_{\mathbf{C}}(A, B)$ for all A, B in $\operatorname{obj}\mathbf{C}$, we say that $\mathbf{C}'$ is a *full subcategory*.

As a trivial example, note that any category is a (full) subcategory of itself.

Examples **1.10**

1. The category **Set** is a subcategory of the category **Rel**, since every map of sets is in particular a relation. But it is not a full subcategory, since it is clear that not every relation is a set map.
2. The category **Ab** is a full subcategory of **Grp**, since every morphism of groups between abelian groups is a morphism of abelian groups.
3. The category **Ring** is a subcategory of **Rng** but not a full subcategory, since for two rings with unity R, S, the constant map $R \to S$ sending all R to the zero element in S is a morphism in **Rng** but not in **Ring**.
4. The category **Haus** is a full subcategory of the category **Top**.
5. Neither of the categories **Grp**, **Ring** is a subcategory of the other.

Given a category $\mathbf{C}$, a full subcategory is completely determined by its class of objects. In this way, we can easily define, for example, the category **FinSet**, which is the full subcategory of **Set** where the class of objects is the class of all finite sets. For any concrete category $\mathbf{C}$, we define then a category **FinC**.

And we can speak of the categories of finitely generated groups, of torsion abelian groups, of noetherian rings, of finite-dimensional vector spaces, of compact Hausdorff topological spaces, and so on. In each case, we first set an ambient category, and then we specify the objects of the full subcategory.

1.10 Definition. We will denote with $\boldsymbol{\Delta}$ the full subcategory of **Poset** such that $\operatorname{obj}\boldsymbol{\Delta} = \{[n] \mid n \in \mathbb{N} \cup \{0\}\}$.

1.5 Special Objects

We now start studying specific properties that an object in a fixed category could have or not have.

1.11 Definition. Let $\mathbf{C}$ be a category. We say that $A \in \operatorname{obj} \mathbf{C}$ is an *initial object* if $\hom_{\mathbf{C}}(A, C)$ has exactly one element for each $C \in \operatorname{obj} \mathbf{C}$. We say that $B \in \operatorname{obj} \mathbf{C}$ is a *final object* if $\hom_{\mathbf{C}}(C, B)$ has exactly one element for each $C \in \operatorname{obj} \mathbf{C}$. Finally, an object which is both an initial and a final object is called a *zero object*.

Examples 1.11

1. In **Set**, the empty set is the only initial object, and the final objects are exactly the sets with only one element. A similar situation happens in **Top**. In particular, there are no zero objects either in **Set** or **Top**.
2. In **Grp**, the trivial group is a zero object.
3. If $\mathbf{P}$ is a category coming from a preordered set P as in example 6 of 1.6, an initial object in $\mathbf{P}$ corresponds to a minimum element of P and a final object in $\mathbf{P}$ with a maximum element of P.

1.12 Definition. Let $\mathbf{C}$ be a category with a zero object. Then in any $\hom_{\mathbf{C}}(A, B)$ there is a well-defined *zero morphism*, $0\colon A \to B$ which is the composition $A \to 0 \to B$. It can be shown that the zero morphism is independent of the choice of the zero object.

Example 1.12

If $G_1, G_2 \in \operatorname{obj} \mathbf{Grp}$, the zero morphism in $\hom_{\mathbf{Grp}}(G_1, G_2)$ is the homomorphism that sends every element of G_1 to the identity.

✎ Exercises 1.2

1. Let $\mathbf{C}$ be a category with a zero object 0. If $0'$ is another zero object, the compositions $A \to 0 \to B$ and $A \to 0' \to B$ are equal.
2. Give an example of a category that has neither an initial object nor a final object.

1.6 Special Morphisms

And now we start studying specific properties that morphisms in a fixed category could have or not have.

1.13 Definition. Let $f\colon A \to B$ be a morphism in a category **C**. We say that f is an *isomorphism* if there is a **C**-morphism $h\colon B \to A$ such that $h \circ f = 1_A$ and $f \circ h = 1_B$. In this case, we say that A and B are *isomorphic*, and we write $A \cong B$.

For example, in the category **Set**, a morphism $f\colon A \to B$ is an isomorphism exactly when it is bijective. Hence in a concrete category, a necessary condition for a morphism to be an isomorphism is that it is bijective.

In our algebraic examples of categories (Examples 1.2), the isomorphisms are exactly the bijective morphisms.

In **Top** and **Haus** isomorphisms are called homeomorphisms. It is well known, however, that it is possible to have a continuous and bijective map $f\colon X \to Y$ without X and Y be homeomorphic. A morphism $f\colon (X, x_0) \to (Y, y_0)$ in $\mathbf{Top}_*$ is an isomorphism if and only if $f\colon X \to Y$ is a homeomorphism. An isomorphism in **Metric** is called an isometry.

Finally, note that in our examples from combinatorics (examples 1.4), in general we need more that a bijective morphism to have an isomorphism.

1.14 Definition. Let $f\colon A \to B$ be a morphism in a category **C**. Then

1. f is *monic* if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$, for any object C and any $g_1, g_2\colon C \to A$.
2. f is *epic* if $h_1 \circ f = h_2 \circ f$ implies $h_1 = h_2$, for any object D and any $h_1, h_2\colon B \to D$.

It is easy to see that in a concrete category **C**, an injective map $f\colon A \to B$ is necessarily monic. For if $g_1, g_2\colon C \to A$ are two **C**-morphisms, then $g_1(c) \neq g_2(c)$ for some $c \in C$ implies that $f\big(g_1(c)\big) \neq f\big(g_2(c)\big)$. Similarly, surjective maps are epic.

1.15 Definition. Let **C** be a category, and A an object in **C**. We define a preorder (reflexive and transitive relation) in the class of monics with codomain A by declaring $f \leq g$ if there is a k such that $f = g \circ k$. We write $f \sim g$ if $f \leq g$ and $g \leq f$. (It then follows that k is an isomorphism). The relation $\sim$ is an equivalence relation and its equivalence classes are called the *subobjects* of A. Similarly, define now a preorder in the class of epics having domain A by $f \geq g$ if there is a k such that $f = k \circ g$, and $f \sim g$ if $f \geq g$ and $g \geq f$. The equivalence classes of $\sim$ are then called the *quotient objects* of A.

✎ Exercises 1.3

1. Isomorphism is an equivalence relation on the class obj **C**.
2. If $f\colon A \to B$ is an isomorphism in a category **C**, the map h in Definition 1.13 is unique.
3. Any two initial objects in a category **C** are isomorphic.

2
Functors

2.1 Definition and Examples

2.1 Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* F from $\mathbf{C}$ to $\mathbf{D}$, denoted $F\colon \mathbf{C} \to \mathbf{D}$, is composed of

1. A map (which we denote by F) from $\operatorname{obj}\mathbf{C}$ to $\operatorname{obj}\mathbf{D}$,
2. For every $A, B \in \operatorname{obj}\mathbf{C}$, a map (also denoted by F) from the set $\hom_{\mathbf{C}}(A, B)$ to the set $\hom_{\mathbf{D}}(FA, FB)$

that satisfy

(a) $F(g \circ f) = F(g) \circ F(f)$ for every pair of morphisms $f, g \in \mathbf{C}$ such that $g \circ f$ is defined,

(b) $F(1_A) = 1_{FA}$ for every $A \in \operatorname{obj}\mathbf{C}$.

So, a functor is a map on objects which preserves compositions and identities. Note that we sometimes omit the parenthesis, and write FA for the image of the object $A \in \operatorname{obj} D$, and similarly for the images of the morphisms.

For our examples, let us begin with general constructions.

Examples **2.1**

1. For every category $\mathbf{C}$, we have the *identity functor* $1_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}$, such that $1_{\mathbf{C}}A = A$ for every $A \in \operatorname{obj}\mathbf{C}$, and $1_{\mathbf{C}}f = f$ for every morphism in $\mathbf{C}$.
2. If $\mathbf{C}'$ is a subcategory of $\mathbf{C}$, there is an *inclusion functor* $\mathbf{C}' \to \mathbf{C}$, which sends every object and every morphism to itself.
3. For any two categories $\mathbf{C}$, $\mathbf{D}$ and object B in $\mathbf{D}$, we have a *constant functor* $F_B\colon \mathbf{C} \to \mathbf{D}$ given by $F_B(A) = B$, $F_B(f) = 1_B$, for all objects A and morphisms f in $\mathbf{C}$.
4. If $\mathbf{C}$ and $\mathbf{D}$ are categories, there is a *projection functor* $p_{\mathbf{C}}\colon \mathbf{C} \times \mathbf{D} \to \mathbf{C}$, sending an object (A, A') of the product to A, and a morphism (f, f') to f.
5. For any category $\mathbf{C}$, we have the *diagonal functor* $\Delta\colon \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ defined by $\Delta(C) = (C, C)$ and $\Delta(f) = (f, f)$.
6. For any comma category $A \downarrow \mathbf{C}$, there is a functor $F\colon A \downarrow \mathbf{C} \to \mathbf{C}$, sending the object $f\colon A \to B$ to B, and a morphism in $A \downarrow \mathbf{C}$ to itself. And similarly, there is a functor from $\mathbf{C} \downarrow A \to \mathbf{C}$,
7. If $\simeq$ is a congruence in $\mathbf{C}$ (see Definition 1.8), there is a *quotient functor* $\mathbf{C} \to \mathbf{C}/{\simeq}$, sending each object to itself and each morphism to its equivalence class.

The next examples are so important that deserve a box by themselves.

Example 2.2

If $\mathbf{C}$ is a concrete category, there is a functor $\mathbf{C} \to \mathbf{Set}$, sending an object A in $\mathbf{C}$, which is a set with structure, to its underlying set. A morphism in $\mathbf{C}$ is sent to the map between the underlying sets. In other words, we just forget about the structure in objects of $\mathbf{C}$, hence this functor is a *forgetful functor*.
Similarly, we have a forgetful functor $\mathbf{Rng} \to \mathbf{Ab}$, given by retaining the additive structure in a ring and forgetting the product, and forgetful functors R-$\mathbf{alg} \to \mathbf{Rng}$, R-$\mathbf{alg} \to R$-$\mathbf{mod}$.

Example 2.3

Let $\mathbf{C}$ be any category, and $A \in \mathbf{C}$ a fixed object. Then there is a functor $\mathbf{C} \to \mathbf{Set}$, denoted as $\hom_{\mathbf{C}}(A, -)$, defined on objects by $B \mapsto \hom_{\mathbf{C}}(A, B)$, and on a $\mathbf{C}$-morphism like $f\colon B \to B'$, by $f_*\colon h \mapsto f \circ h$.
Sometimes the hom sets have extra structure, for example, if A and B are abelian groups, then $\hom_{\mathbf{Ab}}(A, B)$ has an structure of an abelian group and the map f_* is an abelian group map. Hence in this case we get a functor $\hom_{\mathbf{Ab}}(A, -)\colon \mathbf{Ab} \to \mathbf{Ab}$.
We could even go further in another direction: for any category $\mathbf{C}$, there is a functor $\hom_{\mathbf{C}}\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$, sending $(A, B) \to \hom_{\mathbf{C}}(A, B)$ and sending the pair (ϕ, ψ), that is a morphism from (A, B) to (A', B'), to the set map $\hom_{\mathbf{C}}(\phi, \psi)\colon \hom_{\mathbf{C}}(A, B) \to \hom_{\mathbf{C}}(A', B')$ with correspondece rule $h \mapsto \psi \circ h \circ \phi$ (remember that ϕ, being in $\mathbf{C}^{op}$, goes from A' to A when considered in $\mathbf{C}$.)

We now define particular functors, classified by area.

Functors in Algebra

Examples 2.4

1. We define a functor $F\colon \mathbf{Grp} \to \mathbf{Ab}$. For a group G, consider G' be its commutator subgroup, and the quotient G/G', which is an abelian group, so we put $FG = G/G'$. If $f\colon G \to H$ is a group homomorphism, then $f(G')$ is a subgroup of H', so a map $\bar{f}\colon G/G' \to H/H'$ is induced. We define $F(f) = \bar{f}$, it is then straightforward to check it is indeed a functor.
2. The functor $U\colon \mathbf{Ring} \to \mathbf{Grp}$ such that $U(R) =$ the group of invertible elements in R, and if $f\colon R \to S$ is a ring homomorphism, then $U(f)\colon U(R) \to U(S)$ is defined as $U(f) = f|_{U(R)}$.
3. Let $\rho\colon R \to S$ be a morphism of rings. Then we have a functor $\rho^{\#}\colon S\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$ defined the following way: Let N be an S-module. Then $\rho^{\#}N$ has as abelian group the same N, and the action of r is: $rn = \rho(r)n$, where in the right side we have the action of S. Furthermore, $\rho^{\#}(f) = f$.
4. Fix a ring R. Then we have a functor $F\colon \mathbf{Grp} \to R\text{-}\mathbf{alg}$, that sends a group to its group algebra with coefficients in R.

Functors in Topology

Examples 2.5

1. There is a functor $\mathbf{Metric} \to \mathbf{Haus}$, sending each metric space (X, d) to the associated topological space, which is always Hausdorff.
2. We have the so called *loop functor* $\Omega\colon \mathbf{Top}_* \to \mathbf{Top}_*$, whose value at (X, x_0) is the pair $\big(\Omega(X, x_0), c_{x_0}\big)$, where $\Omega(X, x_0) = \{\, f\colon [0,1] \to X \mid f(0) = f(1) = x_0 \,\}$, with the compact-open topology, and c_{x_0} is the constant map with value x_0. If $\phi\colon (X, x_0) \to (Y, y_0)$ is a map in $\mathbf{Top}_*$, then $\Omega(\phi)(f) = \phi \circ f$.
3. There is the *reduced suspension functor* $\Sigma\colon \mathbf{Top}_* \to \mathbf{Top}_*$, whose value at (X, x_0) is composed by the topological space given by the quotient $(X{\times}[0,1])/(X{\times}\{0\}{\cup}X{\times}\{1\}{\cup}\{x_0\}{\times}[0,1])$ with the base point being the corresponding to the collapsed subspace.

Functors in Combinatorics

Examples 2.6

1. There is a functor $K\colon \mathbf{Poset} \to \mathbf{SimplComplex}$, defined on P as the complex with simplices the totally ordered subsets of P P. An order preserving map $f\colon P \to P'$ induces a simplicial map $K(f)\colon K(P) \to K(P')$, given by $K(f)(\{x_0, \ldots, x_n\}) = \{f(x_0), \ldots, f(x_n)\}$.

2. We have a functor $\Delta\colon \mathbf{Graph} \to \mathbf{SimplComplex}$, such that for a graph G, we define $\Delta(G)$ as the simplicial complex with simplices the vertices of complete subgraphs of G. Since any map in **Graph**, say $f\colon G \to G'$, sends complete subgraphs to complete subgraphs, we have a well defined simplicial map $\Delta(f)\colon \Delta(G) \to \Delta(G)$.
3. We have a functor $\mathbf{Graph} \to \mathbf{Graph}$ sending each graph G to its dual graph G^*. This works since a graph map $f\colon G \to G'$ induces a map in the duals $f^*\colon G^* \to (G')^*$.

But by far the most interesting examples are the functors that cross the boundaries of an area of study.

Examples **2.7**

1. We have a functor $F\colon \mathbf{Grp} \to \mathbf{Poset}$, that sends a group G to the set of subgroups of G, partially ordered by containment. A morphism of groups $f\colon G \to H$ sends subgroups to subgroups preserving containment, so $F(f)$ is defined.
2. There is a functor $\pi_1\colon \mathbf{Top}_* \to \mathbf{Grp}$, that sends a pointed topological space to its fundamental group, and a pointed continuous maps to a homomorphism between the corresponding groups.
3. Fixing a ring R, for each $n \in \mathbb{N}\cup\{0\}$ there is a functor $H_n(-, R)\colon \mathbf{Top} \to R\text{-}\mathbf{mod}$, called n-th homology.
4. There is a functor $\mathbf{SimplComplex} \to \mathbf{Top}$, called the geometric realization functor.

Next, let us consider examples that involve small categories.

Examples **2.8**

1. If $\mathbf{C}$ is a discrete category, a functor $F\colon \mathbf{C} \to \mathbf{D}$ is a collection of $\mathbf{D}$-objects, indexed by the objects of $\mathbf{C}$.
2. If $\mathbf{C}$ is the category $\begin{array}{ccc}\bullet & \to & \bullet\\ \downarrow & & \\ \bullet & & \end{array}$, a functor $F\colon \mathbf{C} \to \mathbf{D}$ can be identified with a diagram in $\mathbf{D}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array} \tag{2.1}$$

similarly, if $\mathbf{C}$ is now the category $\begin{array}{ccc} & & \bullet\\ & & \downarrow \\ \bullet & \to & \bullet \end{array}$ or $\bullet \rightrightarrows \bullet$, then a functor $F\colon \mathbf{C} \to \mathbf{D}$ can be identified with

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array} \qquad \text{or} \qquad A \underset{g}{\overset{f}{\rightrightarrows}} B \tag{2.2}$$

respectively

3. Let G_1, G_2 be groups, and $\mathbf{G}_1$ $\mathbf{G}_2$ the corresponding associated categories. Then a functor $F\colon \mathbf{G}_1 \to \mathbf{G}_2$ corresponds to a homomorphism $G_1 \to G_2$.
4. Let G is a group and $\mathbf{G}$ be its associated category. Then a functor $F\colon \mathbf{G} \to \mathbf{Set}$ corresponds to a choice of a set X (the value of $F(*)$), together with a collection of maps $\{g = F(\bullet \xrightarrow{g} \bullet)\colon X \to X\}_{g \in G}$ such that the group operation corresponds to composition, and the neutral element to the identity map. This is exactly the same as having an structure of G-set on X. Hence, for any category $\mathbf{C}$, it seems natural to define a G-object as a functor $F\colon \mathbf{G} \to \mathbf{C}$.

Having defined the concept of functors, we define some more categories.

Examples **2.9**

1. Functors can be composed, this is, if we have categories $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$, and functors $F_1\colon \mathbf{C}_1 \to \mathbf{C}_2$ and $F_2\colon \mathbf{C}_2 \to \mathbf{C}_3$, then there is a functor $F_2 \circ F_1\colon \mathbf{C}_1 \to \mathbf{C}_3$, defined as the composition of the corresponding maps on objects and on morphisms. Moreover, this composition is associative. This, together with the identity functor for any category $\mathbf{C}$ allow us to define the category $\mathbf{SCat}$, with objects the small categories and the set of morphisms between two categories $\mathbf{C}$ and $\mathbf{D}$ is the collection of functors between $\mathbf{C}$ and $\mathbf{D}$. Since a functor $\mathbf{C} \to \mathbf{D}$ is determined by a map $\operatorname{obj}\mathbf{C} \to \operatorname{obj}\mathbf{D}$ and both $\operatorname{obj}\mathbf{C}$, $\operatorname{obj}\mathbf{D}$ are sets, so is $\hom_{\mathbf{SCat}}(\mathbf{C}, \mathbf{D})$. Hence the restriction to small categories is in order to have that the hom sets are, precisely, sets.
2. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor, and A a fixed object of $\mathbf{D}$. We define the category $A \downarrow F$, with objects the pairs $\{(B, f) \mid B \in \operatorname{obj}\mathbf{C}, f \in \hom_{\mathbf{D}}(A, FB)\}$. A morphism from (B, f) to (B', f') is a $\mathbf{C}$-morphism $\phi\colon B \to B'$ such that the diagram

$$\begin{array}{ccc} & & FB \\ & \overset{f}{\nearrow} & \\ A & & \big\downarrow F(\phi) \\ & \underset{f'}{\searrow} & \\ & & FB' \end{array} \qquad (2.3)$$

commutes. In other words, $A \downarrow \mathbf{C}$ is just $A \downarrow 1_{\mathbf{C}}$. We similarly could define a category $F \downarrow A$.

And then we can define more functors.

2.2 Definition. Given a functor $F\colon \mathbf{C} \to \mathbf{D}$, and A a fixed object of $\mathbf{D}$, we have a projection functors $F \downarrow A \to \mathbf{C}$, defined on objects sending (B, f) to B, and on morphisms by sending $\phi\colon (B, f) \to (B', f')$ to ϕ.

We similarly have a projection functor $F \downarrow A \to \mathbf{C}$.

✎ Exercises 2.1

1. Give an example of a category $\mathbf{C}$ and a functor $F\colon \mathbf{C} \to \mathbf{C}$ such that $F(C) = C$ for all $C \in \operatorname{obj}\mathbf{C}$, but $F \neq 1_{\mathbf{C}}$.
2. If we have functors $F\colon \mathbf{C} \to \mathbf{D}$ and $F'\colon \mathbf{C}' \to \mathbf{D}'$, then we can define a functor $(F, F')\colon \mathbf{C} \times \mathbf{C}' \to \mathbf{D} \times \mathbf{D}'$, which in objects is $(F, F')(C) = \big(F(C), F'(C)\big)$.

2.2 Contravariant Functors

The reader would have probably noted that if we fix the second variable in the definition of the hom functor (Example 2.3), we get a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$. This situation deserves a definition.

2.3 Definition. Let $\mathbf{C}$ an $\mathbf{D}$ be categories. A *contravariant functor* $F\colon \mathbf{C} \to \mathbf{D}$ is a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{D}$. That is, F consists of a map from $\operatorname{obj}\mathbf{C}$ to $\operatorname{obj}\mathbf{D}$, and for every $A, B \in \operatorname{obj}\mathbf{C}$, maps from $\hom_{\mathbf{C}}(A, B)$ to $\hom_{\mathbf{D}}(FB, FA)$ such that $F(g \circ f) = F(f) \circ F(g)$ for every morphisms $f, g \in \mathbf{C}$ such that $g \circ f$ is defined, and $F(1_A) = 1_{FA}$ for every $A \in \operatorname{obj}\mathbf{C}$.

The most important examples of a contravariant functor is the following: Let $\mathbf{C}$ be a category, and choose $B \in \operatorname{obj}\mathbf{C}$. We get a contravariant functor $\hom(-, B)\colon \mathbf{C} \to \mathbf{Set}$, such that $\hom(-, B)(A) = \hom(A, B)$, and if $f\colon A \to C$, then

$$\hom(-, B)f\colon \hom(C, B) \to \hom(A, B) \tag{2.4}$$

is given by $g \mapsto g \circ f$. We will denote $\hom(-, B)f$ by f^*.

Example 2.10

We single out one particular case of the previous note. If R is a ring, for any left R-module M one has that $\hom_{R\text{-}\mathbf{mod}}(M, R) = M^*$ has a natural structure of a right R-module, furthermore, if $\phi\colon M \to N$ is a morphism in R-**mod**, then $\phi^*\colon N^* \to M^*$ is a morphism in **mod**-R. Hence we obtain a contravariant functor R-**mod** $\to$ **mod**-R. On the other hand, if M is a right R-module, then $\hom_{\mathbf{mod}\text{-}R}(M, R)$ has a natural structure of left module, and hence in this case we get a contravariant functor **mod**-R $\to$ R-**mod**.

In this context, the functors defined in section 2.1 are called *covariant functors*.

Note that the composition of two contravariant functors is a covariant functor. In particular, composition of the contravariant functors described in Example 2.10 results in a (covariant) functor R-**mod** $\to$ R-**mod**.

2.3 Isomorphism of Categories

2.4 Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. We say that $\mathbf{C}$ is *isomorphic* to $\mathbf{D}$, if there are functors $F\colon \mathbf{C} \to \mathbf{D}$ and $F'\colon \mathbf{D} \to \mathbf{C}$ such that $F' \circ F = 1_{\mathbf{C}}$ and $F \circ F' = 1_{\mathbf{D}}$. We write then $\mathbf{C} \cong \mathbf{D}$.

Examples **2.11**

1. The categories $\mathbb{Z}$-**mod** and **Ab** are isomorphic.
2. If R is commutative, R-**mod** and **mod**-R are isomorphic.
3. The categories $* \downarrow \mathbf{Top}$ and $\mathbf{Top}_*$ are isomorphic.

We can then see that a category is concrete if and only if it is isomorphic to a subcategory of **Set**.

✎ Exercises 2.2

1. For any category $\mathbf{C}$, $(\mathbf{C}^{op})^{op} \cong \mathbf{C}$.
2. For any categories $\mathbf{C}$ and $\mathbf{D}$, $\mathbf{C} \times \mathbf{D} \cong \mathbf{D} \times \mathbf{C}$.
3. For any categories $\mathbf{C}$ and $\mathbf{D}$, $(\mathbf{C} \times \mathbf{D})^{op} \cong \mathbf{C}^{op} \times \mathbf{D}^{op}$.
4. For any categories $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$, $(\mathbf{C} \times \mathbf{D}) \times \mathbf{E} \cong \mathbf{C} \times (\mathbf{D} \times \mathbf{E})$
5. If $\mathbf{C}$ and $\mathbf{D}$ are categories and $F\colon \mathbf{C} \to \mathbf{D}$ is a functor, show that $(A \downarrow F)^{op} \cong F \downarrow A$.
6. If $\mathbf{C}$ and $\mathbf{D}$ are categories and $F\colon \mathbf{C} \to \mathbf{D}$ is a functor, show that if f is an isomorphism in $\mathbf{C}$, then $F(f)$ is an isomorphism in $\mathbf{D}$. If A is an initial object in $\mathbf{C}$, is $F(A)$ an initial object in $\mathbf{D}$?
7. Give an example of a category $\mathbf{C}$ such that $\mathbf{C}$ is isomorphic to $\mathbf{C}^{op}$, and another example when they are not isomorphic.

2.4 Types of Functors

2.5 Definition. Let $F: \mathbf{C} \to \mathbf{D}$ be a functor. We say that F is

1. *faithful*, if for each $A, B \in \operatorname{obj} \mathbf{C}$, the map $F: \hom_{\mathbf{C}}(A, B) \to \hom_{\mathbf{D}}(FA, FB)$ is injective.
2. *full*, if for each $A, B \in \operatorname{obj} \mathbf{C}$, the map $F: \hom_{\mathbf{C}}(A, B) \to \hom_{\mathbf{D}}(FA, FB)$ is surjective.

We note that a subcategory $\mathbf{C}'$ of $\mathbf{C}$ is full if and only if the inclusion functor of $\mathbf{C}'$ on $\mathbf{C}$ is full.

A category $\mathbf{C}$ is concrete if and only if there is a faithful functor $F: \mathbf{C} \to \mathbf{Set}$.

✎ Exercises 2.3

1. Let $F: \mathbf{C} \to \mathbf{D}$ be a full and faithful functor. If $A, B \in \operatorname{obj} \mathbf{C}$ are such that $FA \cong FB$, then $A \cong B$.

3

Natural Transformations

3.1 Definition and Examples

3.1 Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and $F, F' \colon \mathbf{C} \to \mathbf{D}$ be two functors. A *natural transformation* η from F to F', denoted $\eta \colon F \to F'$, is a collection of $\mathbf{D}$-maps $\eta_A \colon FA \to F'A$, one for each $A \in \operatorname{obj} \mathbf{C}$, such that the following diagram commutes for every morphism $f \colon A \to B$ in the category $\mathbf{C}$

$$\begin{array}{ccc} A & \qquad & FA & \xrightarrow{\eta_A} & F'A \\ {\scriptstyle f}\downarrow & & {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F'(f)} \\ B & & FB & \xrightarrow{\eta_B} & F'B \end{array} \tag{3.1}$$

If $\eta \colon F \to F'$ is a natural transformation, we call the $\mathbf{D}$-morphism η_A the *component* of η corresponding to $A \in \operatorname{obj} \mathbf{C}$.

Examples **3.1**

1. Let $F \colon R\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$ be the functor $M \mapsto \hom_{\mathbf{mod}\text{-}R}(\hom_{R\text{-}\mathbf{mod}}(M, R), R)$ considered at the end of section 2.2. We define a natural transformation η from the identity functor to F. Let M be an object in R-**mod**, then we define $\eta_M \colon M \to \hom_{\mathbf{mod}\text{-}R}(M^*, R)$ as $m \mapsto \widehat{m}$,

where $\widehat{m}\colon M^* \to R$ is defined as $\widehat{m}(\phi) = \phi(m)$. We must then check commutativity of

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & \hom_{\mathbf{mod}\text{-}R}(M^*, R) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \hom_{\mathbf{mod}\text{-}R}(f^*, R)} \\ N & \xrightarrow{\eta_N} & \hom_{\mathbf{mod}\text{-}R}(N^*, R) \end{array} \tag{3.2}$$

which follows, since for any $m \in M$ we have that $\widehat{f(m)} = \widehat{m} \circ f^*\colon N^* \to R$.

2. There is a natural isomorphism from the functor $\pi\colon \mathbf{Top}_* \to \mathbf{Grp}$ to the composition of the functors $\mathbf{Top}_* \to \mathbf{Top} \xrightarrow{H_1} \mathbf{Ab} \to \mathbf{Grp}$, given by the Hurewicz homomorphism $\chi_{(X,x_0)}\colon \pi(X, x_0) \to H_1(X)$.

Examples **3.2**

1. Let $\mathbf{I}$ be a discrete category, and two functors $F, F'\colon \mathbf{I} \to \mathbf{C}$ given by the collections of $\mathbf{C}$-objects $\{F_i\}_{i\in\mathrm{obj}\,\mathbf{I}}$, $\{F'_i\}_{i\in\mathrm{obj}\,\mathbf{I}}$ respectively. A natural transformation $\eta\colon F \to F'$ is just a collection of $\mathbf{C}$-morphisms $\{\eta\colon F_i \to F'_i\}_{i\in\mathrm{obj}\,\mathbf{I}}$.

2. Let $\mathbf{I}$ be the category $\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$, and two functors $F, F'\colon \mathbf{I} \to \mathbf{C}$ be represented by the diagrams:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow{\scriptstyle g} & & \\ C & & \end{array} \qquad \begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ \big\downarrow{\scriptstyle g'} & & \\ C' & & \end{array} \tag{3.3}$$

respectively. A natural transformation from F to F' is a collection of three $\mathbf{C}$-morphisms $\phi\colon A \to A'$, $\psi\colon B \to B'$ and $\zeta\colon C \to C'$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & & \\ {\scriptstyle g}\big\downarrow & \searrow^{\phi} & & \searrow^{\psi} & \\ C & & A' & \xrightarrow[f']{} & B' \\ & \searrow_{\zeta} & \big\downarrow{\scriptstyle g'} & & \\ & & C' & & \end{array} \tag{3.4}$$

3. Now let $\mathbf{I}$ be the category $\bullet \rightrightarrows \bullet$, and two functors $F, F'\colon \mathbf{I} \to \mathbf{C}$ be represented by the diagrams:

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \qquad A' \underset{g'}{\overset{f'}{\rightrightarrows}} B' \tag{3.5}$$

A natural transformation from F to F' is then a pair of **D**-morphisms $\phi\colon A \to A'$, $\psi\colon B \to B'$ such that the following squares commute:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ A' & \xrightarrow{f'} & B' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ A' & \xrightarrow{g'} & B' \end{array} \tag{3.6}$$

✎ Exercises 3.1

1. Let **C** and **D** be categories. For each $C \in \operatorname{obj} \mathbf{C}$, define $\sigma_C\colon \mathbf{D} \to \mathbf{C} \times \mathbf{D}$ as $\sigma_C(D) = (C, D)$, and for a **D**-map $f\colon D \to D'$, define $\sigma_C(f)$ as $(1_C, f)$. Show that σ_C is a functor. Then, from a **C**-map $\phi\colon C \to C'$, define a suitable natural transformation $\sigma_f\colon \sigma_C \to \sigma_{C'}$.

3.2 The Functor Category

3.2 ***Proposition.*** *Let F and G be functors $\mathbf{C} \to \mathbf{D}$*

1. *We have that $1_F = \{1_{FA}\}_{A \in \operatorname{obj} \mathbf{C}}$ is a natural transformation from F to F, called the* identity natural transformation *on F.*
2. *Let $\eta\colon F_1 \to F_2$, $\chi\colon F_2 \to F_3$ be natural transformations, where F_1, F_2 and F_3 are functors $\mathbf{C} \to \mathbf{D}$. Then we can define a natural transformation $\chi \circ \eta$ by setting $(\chi \circ \eta)_A = \chi_A \circ \eta_A$. This is called the* composition *of the natural transformations η, χ.*

Now suppose we have **C** and **D** be categories with **C** small, and let F, F' be functors $\mathbf{C} \to \mathbf{D}$. We note that a natural transformation $\eta\colon F \to F'$ is determined by a map

$$\eta\colon \operatorname{obj} \mathbf{C} \to \cup_{C \in \operatorname{obj} \mathbf{C}} \hom_{\mathbf{D}}(FC, F'C). \tag{3.7}$$

Since both $\operatorname{obj} \mathbf{C}$ and $\cup_{C \in \operatorname{obj} \mathbf{C}} \hom_{\mathbf{D}}(FC, F'C)$ are sets, we obtain that the class of all natural transformations from F to F' is actually a set.

Then Proposition 3.2 and this remarks allows us to introduce a new category.

3.3 Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories with $\mathbf{C}$ small. We define a *functor category* $\mathbf{D}^{\mathbf{C}}$ with objects the functors from $\mathbf{C}$ to $\mathbf{D}$, and the morphisms from a functor $F\colon \mathbf{C} \to \mathbf{D}$ to $F'\colon \mathbf{C} \to \mathbf{D}$, the natural transformations from F to F'. The identity and composition given by Proposition 3.2.

3.4 Definition. Remember the category $\boldsymbol{\Delta}$ from Definition 1.10. If $\mathbf{C}$ is any category, then the category of *simplicial objects* in $\mathbf{C}$ is the category $\mathbf{C}^{(\boldsymbol{\Delta}^{op})}$.

3.5 Definition. Let $\mathbf{C}$ and $\mathbf{I}$ be categories with $\mathbf{I}$ small. We have a *diagonal functor* $\Delta\colon \mathbf{C} \to \mathbf{C}^{\mathbf{I}}$, given on objects by $\Delta(C) = F_C$, where $F_C\colon \mathbf{I} \to \mathbf{C}$ is the constant functor of Example 2.1.3.

The set of natural transformations from one functor to another will then be denoted by $\hom_{\mathbf{D}^{\mathbf{C}}}(F, F')$. It will be needed in contexts different that those of functor categories.

3.6 *Proposition.* *Let $\mathbf{C}$ be a small category,*

1. *Let $S\colon \mathbf{D} \to \mathbf{E}$ be a functor. Then there is a functor $S^{\mathbf{C}}\colon \mathbf{D}^{\mathbf{C}} \to \mathbf{E}^{\mathbf{C}}$, given on objects by $S^{\mathbf{C}}(F) = S \circ F$, and if $\eta\colon F \to F'$ is a morphism in $\mathbf{D}^{\mathbf{C}}$, then $S^{\mathbf{C}}(\eta) = S\eta\colon S \circ F \to S \circ F'$ is given by $(S\eta)_A = S(\eta_A)$.*
2. *Let $\mathbf{C}'$ be a small category, and $T\colon \mathbf{C}' \to \mathbf{C}$ be a functor. Then there is a functor $T^*\colon \mathbf{D}^{\mathbf{C}} \to \mathbf{D}^{\mathbf{C}'}$, given on objects by $T^*(F) = F \circ T$, and if $\eta\colon F \to F'$ is a morphism in $\mathbf{D}^{\mathbf{C}}$, then $T^*(\eta) = \eta T$ is given by $(\eta T)_C = \eta_{TC}$.*

3.7 *Proposition.* *Let $\mathbf{C}$ and $\mathbf{I}$ be small categories and $\mathbf{D}$ be an arbitrary category. Then the categories $\mathbf{D}^{\mathbf{C}\times\mathbf{I}}$ and $(\mathbf{D}^{\mathbf{I}})^{\mathbf{C}}$ are isomorphic.*

Proof. Step 1. We define a functor $\Phi\colon \mathbf{D}^{\mathbf{C}\times\mathbf{I}} \to (\mathbf{D}^{\mathbf{I}})^{\mathbf{C}}$. For $S\colon \mathbf{C}\times\mathbf{I} \to \mathbf{D}$, let $\Phi(S)$ be the composition $\mathbf{C} \xrightarrow{\sigma} (\mathbf{C}\times\mathbf{I})^{\mathbf{I}} \xrightarrow{S^{\mathbf{I}}} \mathbf{D}^{\mathbf{I}}$, where σ is the functor of Exercise 3.2.1.

Step 2. We define a functor $\Psi\colon (\mathbf{D}^{\mathbf{I}})^{\mathbf{C}} \to \mathbf{D}^{\mathbf{C}\times\mathbf{I}}$. Let $T\colon \mathbf{C} \to \mathbf{D}^{\mathbf{I}}$. We define $\Psi(T)$ as the composition $\mathbf{C}\times\mathbf{I} \xrightarrow{(T,1_{\mathbf{I}})} \mathbf{D}^{\mathbf{I}}\times\mathbf{I} \xrightarrow{\text{ev}} \mathbf{D}$, where ev is the evaluation functor of Exercise 3.2.2.

Step 3. We check the composition $\Psi \circ \Phi$. For $S\colon \mathbf{C} \times \mathbf{I} \to \mathbf{D}$, we have that $(\Psi \circ \Phi)(S) = \operatorname{ev} \circ (S^{\mathbf{I}} \circ \sigma, 1_{\mathbf{I}})$. Evaluating at $(C, i) \in \operatorname{obj} \mathbf{C} \times \mathbf{I}$, we obtain

$$[\operatorname{ev} \circ (S^{\mathbf{I}} \circ \sigma, 1_{\mathbf{I}})](C, i) = \operatorname{ev}\big((S^{\mathbf{I}} \circ \sigma)(C), i\big) \tag{3.8}$$

$$= [(S^{\mathbf{I}} \circ \sigma)(C)](i) \tag{3.9}$$

$$= (S \circ \sigma_C)(i) = S(C, i) \tag{3.10}$$

and so $(\Psi \circ \Phi)(S) = S$.

Step 4. We check the composition $\Phi \circ \Psi$. For $T\colon \mathbf{C} \to \mathbf{D}^{\mathbf{I}}$, we have that $(\Phi \circ \Psi)(T) = \big(\operatorname{ev} \circ (T, 1_{\mathbf{I}})\big)^{\mathbf{I}} \circ \sigma$. Evaluating at $C \in \operatorname{obj} \mathbf{C}$, we get

$$[\big(\operatorname{ev} \circ (T, 1_{\mathbf{I}})\big)^{\mathbf{I}} \circ \sigma](C) = \big(\operatorname{ev} \circ (T, 1_{\mathbf{I}})\big) \circ \sigma_C\colon \mathbf{I} \to \mathbf{D} \tag{3.11}$$

and evaluating this at $i \in \operatorname{obj} \mathbf{I}$, we get

$$[\big(\operatorname{ev} \circ (T, 1_{\mathbf{I}})\big) \circ \sigma_C](i) = \big(\operatorname{ev} \circ (T, 1_{\mathbf{I}})\big)(C, i) \tag{3.12}$$

$$= \operatorname{ev}\big(T(C), i\big) = T(C)(i) \tag{3.13}$$

hence $(\Phi \circ \Psi)(T) = T$.

Step 5. We leave as an exercise to prove that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the corresponding identity functors. □

✎ Exercises 3.2

1. For categories $\mathbf{C}$ and $\mathbf{D}$, with $\mathbf{D}$ small, we get a functor $\sigma\colon \mathbf{C} \to (\mathbf{C} \times \mathbf{D})^{\mathbf{D}}$, $C \mapsto \sigma_C$, where σ_C is as in Exercise 3.1.1.
2. For categories $\mathbf{C}$ and $\mathbf{D}$, with $\mathbf{C}$ small, we have an *evaluation functor* $\operatorname{ev}\colon \mathbf{D}^{\mathbf{C}} \times \mathbf{C} \to \mathbf{D}$, defined on objects by $\operatorname{ev}(F, C) = F(C)$.
3. Complete the proof of Proposition 3.7.1
4. If $\mathbf{D}$ is small, then $\mathbf{C}^{(\mathbf{D}^{\mathrm{op}})} \cong (\mathbf{C}^{\mathrm{op}})^{\mathbf{D}} \cong (\mathbf{C}^{\mathbf{D}})^{\mathrm{op}}$.
5. Does the map $\operatorname{obj} \mathbf{D}^{\mathbf{E}} \to \operatorname{obj}(\mathbf{E}^{\mathbf{C}})^{\mathbf{D}^{\mathbf{C}}}$ given by $S \mapsto S^{\mathbf{C}}$ from Proposition 3.6.1 define a functor?

3.3 Equivalence of Categories

3.8 Definition. Let F, F' be functors $\mathbf{C} \to \mathbf{D}$, and $\eta \colon F \to F'$ a natural transformation. If all components η_A are isomorphisms, we say that the functors F and F' are *naturally isomorphic*. We denote this as $F \cong F'$.

Natural isomorphism arise frequently in the following way: one observes an isomorphism $FA \cong F'A$ which is defined independently of the object A. One then says that the isomorphism is natural on A.

Note that if $\mathbf{C}$ is a small category, then two functors $F, F' \colon \mathbf{C} \to \mathbf{D}$ are naturally isomorphic if and only if there are isomorphic objects in the functor category $\mathbf{D}^{\mathbf{C}}$.

3.9 Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. We say that $\mathbf{C}$ is *equivalent* to $\mathbf{D}$, if there are functors $F \colon \mathbf{C} \to \mathbf{D}$ and $F' \colon \mathbf{D} \to \mathbf{C}$ such that $F' \circ F \cong 1_{\mathbf{C}}$ and $F \circ F' \cong 1_{\mathbf{D}}$.

3.4 The Yoneda Lemma

3.10 Definition. Let $\mathbf{C}$ be a category. We say that a functor $F \colon \mathbf{C} \to \mathbf{Set}$ is *representable* if there is $A \in \operatorname{obj} \mathbf{C}$ such that $F \cong \hom_{\mathbf{C}}(A, -)$.

3.11 *Theorem.* (Yoneda Lemma) Let $\mathbf{C}$ be any category, $F \colon \mathbf{C} \to \mathbf{Set}$ a functor, and $A \in \operatorname{obj} \mathbf{C}$. Then there is a bijection $FA \leftrightarrow \hom_{\mathbf{Set}^{\mathbf{C}}}\big(\hom_{\mathbf{C}}(A, -), F\big)$.

Proof. Given $a \in A$, we define a natural transformation $\eta(a) \colon \hom_{\mathbf{C}}(A, -) \to F$. Its component at $B \in \operatorname{obj} \mathbf{C}$ is $\eta(a)_B \colon \hom_{\mathbf{C}}(A, B) \to FB$ sending $f \mapsto F(f)(a)$. To see that $\eta(a)$ is actually a natural transformation, we check commutativity of the diagram:

$$\begin{array}{ccccc} B & \qquad & \hom_{\mathbf{C}}(A, B) & \xrightarrow{\eta(a)_B} & FB \\ {\scriptstyle \phi}\downarrow & & {\scriptstyle \hom_{\mathbf{C}}(A,\phi)}\downarrow & & \downarrow{\scriptstyle F(\phi)} \\ B' & & \hom_{\mathbf{C}}(A, B') & \xrightarrow{\eta(a)_{B'}} & FB' \end{array} \tag{3.14}$$

And that follows, since evaluating at $f \in \hom_{\mathbf{C}}(A, B)$ it comes down to $F\phi\big(F(f)(a)\big) = F(\phi \circ f)(a)$. Hence we have defined a set map $\eta \colon FA \to \hom_{\mathbf{C}}(A, -)$. For the inverse, we propose the map $\hom_{\mathbf{C}}(A, -) \to FA$ given by $\kappa \colon \kappa_A(1_A)$. We now show that both maps are actually inverses to each other.

First, given $a \in FA$, we have that $\eta(a)_A(1_A) = F(1_A)(a) = 1_{FA}(a) = a$.

Then, given $\kappa \in \hom_{\mathbf{Set}^{\mathbf{C}}}\big(\hom_{\mathbf{C}}(A,-), F\big)$, we want to show that $\eta(\kappa_A(1_A)) = \kappa$. Let $B \in \operatorname{obj} \mathbf{C}$. We use commutativity of

$$\begin{array}{ccccc} A & \qquad & \hom_{\mathbf{C}}(A,A) & \xrightarrow{\eta_A} & FA \\ {\scriptstyle f}\downarrow & & {\scriptstyle \hom_{\mathbf{C}}(A,f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ B & & \hom_{\mathbf{C}}(A,B) & \xrightarrow{\eta_B} & FB \end{array} \tag{3.15}$$

since $\eta\big(\kappa_A(1_A)\big)_B(f) = F(f)\big(\kappa_A(1_A)\big) = \big(F(f) \circ \eta_A\big)(1_A) = \big(\eta_B \circ \hom_{\mathbf{C}}(A,f)\big)(1_A) = \kappa_B(f)$. This finishes the proof. □

Keeping the hypothesis of the Yoneda Lemma, if A' is a $\mathbf{C}$-object, then applying said Lemma to $F = \hom_{\mathbf{C}}(A',-)$, we get a bijection between the set $\hom_{\mathbf{C}}(A',A)$, and the set of natural transformations from $\hom_{\mathbf{C}}(A,-)$ and $\hom_{\mathbf{C}}(A',-)$. We even have the following:

3.12 *Corollary.* *Let A and A' be $\mathbf{C}$-objects. Then the functors $\hom_{\mathbf{C}}(A,-)$ and $\hom_{\mathbf{C}}(A',-)$ are naturally isomorphic if and only if A and A' are isomorphic.*

Proof. We apply the Yoneda Lemma with $F = \hom_{\mathbf{C}}(A',-)$.

Let $\phi \in FA = \hom_{\mathbf{C}}(A',A)$ an isomorphism. As in the proof of the Yoneda Lemma, we have a natural transformation $\eta(\phi)$ with component at $B \in \operatorname{obj} B$ being the map $\eta(\phi)_B \colon \hom_{\mathbf{C}}(A,B) \to \hom_{\mathbf{C}}(A',B)$ which sends $f \mapsto f \circ \phi$. This is a bijection for every $B \in \operatorname{obj} \mathbf{C}$ (its inverse is $\eta(\phi^{-1})_B$), so the functors mentioned are naturally isomorphic.

Now, let $\eta\colon \hom_{\mathbf{C}}(A,-) \to \hom_{\mathbf{C}}(A',-)$ be a natural isomorphism. Let $\phi = \eta_A(1_A)$, we will show that ϕ is an isomorphism. Consider the following diagram:

$$\begin{array}{ccccc} A' & \qquad & \hom_{\mathbf{C}}(A,A') & \xrightarrow{\eta_{A'}} & \hom_{\mathbf{C}}(A',A') \\ {\scriptstyle \phi}\downarrow & & {\scriptstyle \phi_*}\downarrow & & \downarrow{\scriptstyle \phi_*} \\ A & & \hom_{\mathbf{C}}(A,A) & \xrightarrow{\eta_A} & \hom_{\mathbf{C}}(A',A) \end{array} \tag{3.16}$$

Since $\eta_{A'}$ is a bijection, there is an $f\colon A \to A'$ such that $\eta_{A'}(f) = 1_{A'}$. The diagram (3.16) then says that $\eta_A(\phi \circ f) = \phi$. But also $\eta_A(1_A) = \phi$, so that $\phi \circ f = 1_A$.

To prove now that $f \circ \phi = 1_{A'}$, we consider the diagram

$$\begin{array}{ccccc} A & & \hom_{\mathbf{C}}(A, A) & \xrightarrow{\eta_A} & \hom_{\mathbf{C}}(A', A) \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ A' & & \hom_{\mathbf{C}}(A, A') & \xrightarrow{\eta_{A'}} & \hom_{\mathbf{C}}(A', A') \end{array} \tag{3.17}$$

since evaluating at 1_A, we get that $f \circ \phi = \eta_{A'}(f) = 1_{A'}$. □

Exercises 3.3

1. A contravariant functor F from $\mathbf{C}$ to $\mathbf{Set}$ is defined to be representable if there is $B \in \operatorname{obj} \mathbf{C}$ such that $F \cong \hom_{\mathbf{C}}(-, B)$. State and prove a Yoneda Lemma for contravariant functors.
2. Let $\mathbf{C}$ be any category, and consider the functor $\hom_{\mathbf{C}} \colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$ of Example 2.3. Using Proposition 3.7, to that functor corresponds a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathbf{C}}$. Show that this last functor full, faithful, and injective in objects. (Note that this, together with Exercise 2.3.1, gives another way to prove Corollary 3.12).

Part II

Limits

4

Limits and Colimits

In this chapter, **I** will always denote a small category. And, for a functor $F\colon \mathbf{I} \to \mathbf{C}$ and $i \in \operatorname{obj} \mathbf{I}$, we will denote the object $F(i)$ by F_i.

4.1 Limits

4.1 Definition. Let $F\colon \mathbf{I} \to \mathbf{C}$ be a functor and Y an object in $\mathbf{C}$.

1. A *source* from Y to F is a collection of $\mathbf{C}$-maps $\{\delta_i : Y \to F_i\}$, one for each $i \in \operatorname{obj} \mathbf{I}$.
2. A *natural source* from Y to F is a source from Y to F such that the following diagram commutes for all objects i, j in $\mathbf{I}$ and maps $m\colon i \to j$.

$$\begin{array}{ccc} & Y & \\ {}^{\delta_i}\swarrow & & \searrow^{\delta_j} \\ F_i & \xrightarrow{F(m)} & F_j \end{array} \tag{4.1}$$

Examples 4.1

1. If A is an initial object in $\mathbf{C}$, then for each $C \in \operatorname{obj} \mathbf{C}$ there is a unique map $\delta_C : A \to C$. The collection of maps $\{\delta_C\}_{C \in \operatorname{obj} \mathbf{C}}$ is a natural source from A to $1_{\mathbf{C}}$.

2. If the category **I** is discrete, then any source is a natural source.

3. If **I** is the category $\begin{array}{cc} & \bullet \\ & \downarrow \\ \bullet \longrightarrow & \bullet \end{array}$, we saw that a functor $F\colon \mathbf{I} \to \mathbf{C}$ can be identified with a diagram in **C**

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array} \tag{4.2}$$

A natural source from Y to F may be identified with a pair of morphisms $\phi\colon Y \to A$, $\psi\colon Y \to B$ making commute the following square:

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & A \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array} \tag{4.3}$$

4. Let **I** be the category $\bullet \rightrightarrows \bullet$. A natural source to the functor $F\colon \mathbf{I} \to \mathbf{C}$

$$A \underset{g}{\overset{f}{\rightrightarrows}} B\ . \tag{4.4}$$

can be identified with a map $k\colon Y \to A$ such that $f \circ k = g \circ k$.

4.2 Definition. Let $F\colon \mathbf{I} \to \mathbf{C}$ be a functor. A *limiting source* of F is a natural source from some **C**-object Y to F, say $\{\delta_i\colon Y \to F_i\}$, such that for any other natural source $\{\delta'_i\colon Y' \to F_i\}$ there is a unique **C**-morphism $M\colon Y' \to Y$ such that the diagram

$$\begin{array}{ccc} Y' & \overset{M}{\dashrightarrow} & Y \\ & {\scriptstyle \delta'_i}\searrow \quad \swarrow{\scriptstyle \delta_i} & \\ & F_i & \end{array} \tag{4.5}$$

commutes for all $i \in \operatorname{obj}\mathbf{I}$. In this case, we say that X is a limit of F.

Note that any two limits of F are isomorphic. This limit is sometimes called inverse limit or projective limit.

Examples **4.2**

1. Let p be a prime. The ring of p-*adic integers* $\mathbb{Z}_p$ can be defined formally as the set of sums of the form $r_0 + r_1p + r_2p^2 + \cdots$, with $0 \le r_i < p$, and with the natural[1] sum and multiplication.

A functor $F\colon \mathbf{N}^{op} \to \mathbf{Ab}$ is determined by a collection of abelian groups $F(n) = A_n$, one for each $n \in \mathbb{N}$ and morphisms $A_m \to A_n$ whenever $n \leq m$. Let $A_n = \mathbb{Z}/p^n$ and $A_m \to A_n$ be determined by $\overline{1} \mapsto \overline{1}$. We claim that $\lim F = \mathbb{Z}_p$. We have maps $\eta_n\colon \mathbb{Z}_p \to \mathbb{Z}/p^n$ defined by $r_0 + r_1p + r_2p^2 + \cdots \mapsto r_0 + r_1p + r_2p^2 + \cdots + r_{n-1}p^{n-1}$, and they make the corresponding diagram commute. If we have a collection of maps $\eta'_n\colon X \to \mathbb{Z}/p^n$ that make commute diagram [2], then we can define a map $X \to \mathbb{Z}_p$ the following way: By commutativity, given $x \in X$ we can find integers r_i such that $0 \leq r_i < p$ and such that $\eta'_n(x) = \overline{r_0 + r_1p + r_2p^2 + \cdots + r_{n-1}p^{n-1}}$ for all $n \in \mathbb{N}$. Define the map $X \to \mathbb{Z}_p$ as $x \to r_0 + r_1p + r_2p^2 + \cdots$. This has the commutativity and uniqueness properties required for the limit.

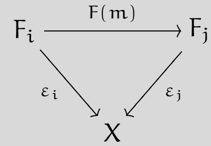
Exercises 4.1

1. Let $F\colon \mathbf{I} \to \mathbf{C}$ be a functor, $\{\delta_i\colon Y \to F_i\}$ a natural source, and $\phi\colon Y' \to Y$ a $\mathbf{C}$-map. Then $\{\delta_i \circ \phi\colon Y' \to F_i\}$ is a natural source.
2. Let $F\colon \mathbf{I} \to \mathbf{C}$ be a functor, $\{\delta_i\colon Y \to F_i\}$ a limiting source, and $\phi, \psi\colon Y' \to Y$ $\mathbf{C}$-maps such that $\delta_i \circ \phi = \delta_i \circ \psi$ for all $i \in \operatorname{obj} \mathbf{I}$. Then $\phi = \psi$.
3. Let $F\colon \mathbf{I} \to \mathbf{C}$ be a functor, $\{\delta_i\colon Y \to F_i\}$ a limiting source, $\{\delta'_i\colon Y' \to F_i\}$ a natural source and $M\colon Y' \to Y$ a $\mathbf{C}$-map such that $\delta_i \circ M = \delta'_i$ for all $i \in \operatorname{obj} \mathbf{I}$. Show that $\{\delta'_i\colon Y' \to F_i\}$ is a limiting source if and only if M is an isomorphism.

4.2 Colimits

4.3 Definition. Let $F\colon \mathbf{I} \to \mathbf{C}$ be a functor and X an object in $\mathbf{C}$.

1. A *sink* from F to X is a collection of $\mathbf{C}$-maps $\varepsilon_i\colon F_i \to X$, one for each $i \in \operatorname{obj} \mathbf{I}$.
2. A sink from F to X is *natural* if the following diagram commutes for all objects i, j in $\mathbf{I}$ and maps $m\colon i \to j$.



(4.6)

In other words, a sink from F to X is just a collection of objects in the comma category $F\downarrow X$, indexed by $\operatorname{obj}\mathbf{I}$. If we denote by π the projection functor $F\downarrow X \to \mathbf{I}$, a natural sink can be identified with a functor $\theta\colon \mathbf{I} \to F\downarrow X$ such that $\pi \circ \theta = 1_{\mathbf{I}}$. Another way that we can think of a sink from F to X is as a natural transformation from F to $\Delta(X)$.

Examples **4.3**

1. If B is a final object in $\mathbf{C}$, then for each $C \in \operatorname{obj}\mathbf{C}$ there is a unique map $\eta_C\colon C \to B$, thus we get a sink from $1_{\mathbf{C}}$ to B. The uniqueness of the maps going to B shows that the sink is natural.
2. If the category $\mathbf{I}$ is discrete, then any sink is a natural sink.
3. Let $\mathbf{I}$ be the category category $\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$, and the functor $F\colon \mathbf{I} \to \mathbf{C}$ be represented by the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array} \tag{4.7}$$

A natural sink from F to X may be identified with a pair of morphisms $\phi\colon B \to X$, $\psi\colon C \to X$ making commute the following square:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \phi} \\ C & \xrightarrow{\psi} & X \end{array} \tag{4.8}$$

4. Let $\mathbf{I}$ be the category $\bullet \rightrightarrows \bullet$, and F be the functor:

$$A \underset{g}{\overset{f}{\rightrightarrows}} B\,. \tag{4.9}$$

Then a natural source from the functor $F\colon \mathbf{I} \to \mathbf{C}$ to X can be identified with a map $l\colon B \to X$ such that $l \circ f = l \circ g$.

4.4 Definition. Let $F\colon \mathbf{I} \to \mathbf{C}$ be a functor. A *limiting sink* for F is a natural sink from F to some $\mathbf{C}$-object X, say $\{\varepsilon_i\colon F_i \to X\}$, such that for any other natural sink $\{\varepsilon_i'\colon F_i \to X'\}$, there is a unique map $M\colon X \to X'$ such that the diagram

$$\begin{array}{ccc} & F_i & \\ {\scriptstyle \varepsilon_i}\swarrow & & \searrow{\scriptstyle \varepsilon_i'} \\ X & \overset{M}{\dashrightarrow} & X' \end{array} \tag{4.10}$$

commutes for all $i \in \operatorname{obj} \mathbf{I}$. In this case, we say that X is a *colimit* of F, and we denote it by $\operatorname{colim} F$.

Note that any two colimits of F are isomorphic.

The colimit is also sometimes called direct limit or inductive limit, in which case it is denoted as $\varinjlim F$.

A functor $F\colon \mathbf{I} \to \mathbf{C}$ is called an $\mathbf{I}$-*diagram* in $\mathbf{C}$.

Examples 4.4

1. Let $F\colon \mathbf{I} \to \mathbf{Set}$ be a functor. Let $U = \{ (i, x) \mid i \in I, x \in F(i) \}$. Then $\operatorname{colim} F = U/{\sim}$, where $\sim$ is the equivalence relation generated by $(i, x) \sim \big(i', F(m)(x)\big)$, where $m\colon i \to i'$ is a morphism of $\mathbf{I}$.
2. We show that any module M is the colimit of its finitely generated submodules. For any finitely generated submodule N we have an inclusion map $\rho_N\colon N \to M$. They commute with all inclusions $N_1 \le N_2$, so we have a natural sink. Suppose now that we have a module X and maps $\rho'_N\colon N \to X$, one for each finitely generated submodule N, forming another natural sink. We need to define an $f\colon M \to X$. Let $m \in M$, and define $f(m) = \rho'_N(m)$, where N is any finitely generated submodule containing m. This is well defined by commutativity, and it is a module homomorphism with the desired properties.
3. Let $\mathbb{N}$ be the partially ordered set of positive integers where we set $n \le m$ whenever $n|m$. Let $\mathbf{N}$ be the category associated to such poset. We define a functor $F\colon \mathbf{N} \to \mathbf{Ab}$ by $F(n) = \mathbb{Z}/n$ and if $n \le m$, we set $F(n) \to F(m)$ to be determined by $\overline{1} \to \overline{\frac{m}{n}}$. One has then to prove that F is actually a functor. We claim then $\operatorname{colim} F = \mathbb{Q}/\mathbb{Z}$. We have maps $\varepsilon_n\colon \mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$ defined by $\overline{1} \mapsto \overline{\frac{1}{n}}$, and they form a natural sink from F to $\mathbb{Q}/\mathbb{Z}$. Now, if we have another natural sink from F $\varepsilon'_n\colon \mathbb{Z}/n \to X$ then we can define a map $\mathbb{Q}/\mathbb{Z} \to X$ by $\overline{\frac{p}{q}} \mapsto \varepsilon'_q(\overline{p})$, that makes the corresponding diagram commute and it is unique with respect to such property.
4. We have a functor $\mathbf{I}\downarrow -\colon \mathbf{I} \to \mathbf{SCat}$ by sending i to $\mathbf{I}\downarrow i$. We calculate $\operatorname{colim} \mathbf{I}\downarrow -$. For $i \in \operatorname{obj} \mathbf{I}$, let $\varepsilon_i\colon \mathbf{I}\downarrow i \to \mathbf{I}$ be the projection functor of Definition 2.2. Then the ε_i form a limiting cone for $\mathbf{I}\downarrow -$, hence $\operatorname{colim} \mathbf{I}\downarrow - = \mathbf{I}$. (See [BK72, XI, 2.3])

Exercises 4.2

1. Let $\mathbf{C}$ be a small category, and $C \in \operatorname{obj} \mathbf{C}$. Show that the colimit of the functor $\hom_{\mathbf{C}}(C, -)\colon \mathbf{C} \to \mathbf{Set}$ is the one-point set.

4.3 More Examples

Some particular cases of limits and colimits have special names, we consider them here.

Examples **4.5**

1. If $\mathbf{I}$ is a discrete category, then a functor $F\colon \mathbf{I} \to \mathbf{C}$ can be identified with a collection $\{F_i\}_{i \in \operatorname{obj} \mathbf{I}}$ of objects in $\mathbf{C}$. Then the colimit of F is called the *coproduct* of the F_i, denoted $\coprod F_i$. However, if the set $\mathbf{I}$ is finite, say $\mathbf{I} = \{1, \dots, n\}$ we denote $\coprod F_i$ as $F_1 \sqcup \cdots \sqcup F_n$. Dually, the limit of F is called the *product* of the F_i and is denoted $\prod F_i$. It is denoted as $F_1 \times \cdots \times F_n$ if $\mathbf{I}$ is finite.
2. There is only one functor $\mathbf{0} \to \mathbf{C}$, and its colimit is an initial object in $\mathbf{C}$. Dually, its limit is a final object in $\mathbf{C}$.
3. Considering the identity functor $1_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}$, we have that $\lim 1_{\mathbf{C}}$ exists if and only if $\mathbf{C}$ has a final object (which is then $\lim 1_{\mathbf{C}}$). Similarly, $\operatorname{colim} 1_{\mathbf{C}}$ exists if and only if $\mathbf{C}$ has an initial object.
4. If $\mathbf{I}$ is the category $\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$, a limit of a functor $F\colon \mathbf{I} \to \mathbf{C}$ is called the *pullback* of the corresponding diagram. Dually, if $\mathbf{I}$ is the category $\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$, a limit of a functor $F\colon \mathbf{I} \to \mathbf{C}$ is called the *pushout* of the corresponding diagram.
5. Now let $\mathbf{I}$ be the category $\bullet \rightrightarrows \bullet$. The limit of the functor $F\colon \mathbf{I} \to \mathbf{C}$
$$A \underset{g}{\overset{f}{\rightrightarrows}} B\,. \tag{4.11}$$
can be identified (see Example 4.1) with a certain $\mathbf{C}$-map $K \to A$, and it is called the *equalizer* of f and g. Dually, the colimit of F can be identified with certain map $B \to C$ and is called the *coequalizer* of f and g.
6. An important special case of the last construction is the following: Let $\mathbf{C}$ be a category with a zero object 0, and $f\colon A \to B$ a morphism in $\mathbf{C}$. Then we define the *cokernel* of f, denoted $\operatorname{coker} f$ as the coequalizer of f and $0\colon A \to B$. Also we define the *kernel* of f, $\ker f$, as the equalizer of f and 0.

Exercises 4.3

1. A sink from $F\colon \mathbf{I} \to \mathbf{C}$ to X can be identified with a $\mathbf{C}$-morphism $\coprod_{i \in \operatorname{obj} \mathbf{I}} F_i \to X$.
2. If B is a final object in a category $\mathbf{C}$ with finite products, then $C \times B \cong C$ for all objects C in $\mathbf{C}$.
3. If $\mathbf{C}$ is a category with finite products and C, D are objects in $\mathbf{C}$, then $C \times D \cong D \times C$.

4. If $\mathbf{C}$ is a category with finite products and C, D, E are objects in $\mathbf{C}$, then $(C \times D) \times E \cong C \times D \times E \cong C \times (D \times E)$.
5. Coequalizers are epics.
6. If $\mathbf{C}$ is a category with finite products and C is an object in $\mathbf{C}$, then we get a functor $C \times -\colon \mathbf{C} \to \mathbf{C}$, sending $A \mapsto C \times A$.

4.4 Limit and Colimit as Functors

4.5 Definition. We say that the category $\mathbf{C}$ is *cocomplete* if $\operatorname{colim} F$ exists for any functor $F\colon \mathbf{I} \to \mathbf{C}$ and any $\mathbf{I}$ a small category, and that $\mathbf{C}$ is *complete* if $\lim F$ exists for any functor $F\colon \mathbf{I} \to \mathbf{C}$ and any $\mathbf{I}$ a small category.

Remember that a sink from F to X can be identified with a natural transformation from F to $\Delta(X)$. With this viewpoint, we obtain the following result: if $\operatorname{colim} F$ exists, then any natural transformation from F to $\Delta(X)$ induces a unique map $\operatorname{colim} F \to X$ such that the following diagram in $\mathbf{C}^{\mathbf{I}}$ commutes:

$$\begin{array}{ccc} & F & \\ \swarrow & & \searrow \\ \Delta(\operatorname{colim} F) & \dashrightarrow & \Delta(X) \end{array} \tag{4.12}$$

Similarly, if $\lim F$ exists, then any natural transformation from $\Delta(Y)$ to F induces a map $Y \to \lim F$ such that the following diagram commutes:

$$\begin{array}{ccc} \Delta(Y) & \dashrightarrow & \Delta(\lim F) \\ \searrow & & \swarrow \\ & F & \end{array} \tag{4.13}$$

Hence, if $\mathbf{C}$ is cocomplete and $\mathbf{I}$ is small, we obtain a functor $\operatorname{colim}\colon \mathbf{C}^{\mathbf{I}} \to \mathbf{C}$, defined on the object F to be $\operatorname{colim} F$, and if $\eta\colon F \to F'$ is a natural transformation, we get a map $\operatorname{colim} \eta\colon \operatorname{colim} F \to \operatorname{colim} F'$ induced by the natural transformation $F \to F' \to \Delta(\operatorname{colim} F')$. The map $\operatorname{colim} \eta$ is also characterized as the only $\mathbf{D}$-map that makes the following square commute:

$$\begin{array}{ccc} F_i & \xrightarrow{\eta_i} & F'_i \\ \varepsilon_i \downarrow & & \downarrow \varepsilon'_i \\ \operatorname{colim} F & \overset{\operatorname{colim} \eta}{\dashrightarrow} & \operatorname{colim} F' \end{array} \tag{4.14}$$

for all $i \in \operatorname{obj} \mathbf{I}$.

Similarly if $\mathbf{C}$ is complete and $\mathbf{I}$ is small, we can define a functor $\lim\colon \mathbf{C}^{\mathbf{I}} \to \mathbf{C}$, sending F to $\lim F$, and if $F \to F'$ is a natural transformation, we get a map $\lim F \to \lim F'$ induced by the natural transformation $\Delta(\lim F) \to F \to F'$.

Another useful remark is that if $T\colon \mathbf{C} \to \mathbf{D}$ is a functor, and $A \in \operatorname{obj} \mathbf{C}$, then the composition

$$\mathbf{I} \xrightarrow{\Delta(A)} \mathbf{C} \xrightarrow{T} \mathbf{D} \tag{4.15}$$

that is $T\big(\Delta(A)\big)$, is equal to $\Delta(TA)$.

✎ Exercises 4.4

1. Let $\mathbf{C}$ be a cocomplete category and $\mathbf{I}$ be a small category. Form the coproducts $\coprod_{\phi\colon X \to Y} F(X)$, $\coprod_{X \in \operatorname{obj} \mathbf{C}} F(X)$, where $\phi\colon X \to Y$ varies over all $\mathbf{I}$-morphisms. Using 4.14, construct maps $\coprod_{\phi} F(\phi)\colon \coprod_{\phi} F(X) \to \coprod_X F(X)$ and $\coprod_{\phi} 1_{F(X)}\colon \coprod_{\phi} F(X) \to \coprod_X F(X)$. Show that the coequalizer of these two maps is $\operatorname{colim} F$.

4.5 Preservation of Limits

Let $F\colon \mathbf{I} \to \mathbf{C}$, $T\colon \mathbf{C} \to \mathbf{D}$ be functors, and Y an object in $\mathbf{C}$. If $\{\delta_i\colon Y \to F_i\}$ is a source from Y to F, then $\{T \circ \delta_i\colon T(Y) \to T(F_i)\}$ is a source from $T(Y)$ to $T \circ F$. Furthermore, if the original source is natural, then the resulting source is natural as well.

Suppose then that $\{\delta_i\colon Y \to F_i\}$ is a limiting source, that is, $Y = \lim F$. Since $\{T \circ \delta_i\}$ is a natural source from $T(Y)$ to $T \circ F$, by the definition of limit there is a unique map $M\colon T(\lim F) \to \lim(T \circ F)$ making the following triangle commute for all $i \in \operatorname{obj} \mathbf{I}$.

$$\begin{array}{ccc} T(\lim F) & \overset{M}{\dashrightarrow} & \lim(T \circ F) \\ & {\scriptstyle T\circ\delta_i} \searrow \quad \swarrow & \\ & T \circ F & \end{array} \tag{4.16}$$

The following definition considers the case in which the map M is an isomorphism.

4.6 Definition. We say that the functor $\mathsf{T}\colon \mathbf{C} \to \mathbf{D}$ *preserves limits*, if for any functor $\mathsf{F}\colon \mathbf{I} \to \mathbf{C}$ and limiting source $\{\delta_i\colon \mathsf{Y} \to \mathsf{F}_i\}$ from Y to F, then $\{\mathsf{T} \circ \delta_i\colon \mathsf{T}(\mathsf{Y}) \to \mathsf{T}(\mathsf{F}_i)\}$ is a limiting source from $\mathsf{T}(\mathsf{Y})$ to $\mathsf{T} \circ \mathsf{F}$.

Theorem 5.6 will give a condition on F that ensures preservation of limits.

5

Universals and Adjoints

5.1 Universals

5.1 Definition. Let **C** and **D** be categories, $F\colon \mathbf{C} \to \mathbf{D}$ be a functor, and $B \in \operatorname{obj} \mathbf{D}$. Then a *universal* from B to F is a pair (U, u) where $U \in \operatorname{obj} \mathbf{C}$ and $u\colon B \to FU$ is a **D**-map, such that if $h\colon B \to FU'$ is any **D**-map with $U' \in \operatorname{obj} \mathbf{C}$, then there is a unique **C**-map $m\colon U \to U'$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & FU & \quad & U \\ & \overset{u}{\nearrow} & & & \\ B & & \big\downarrow F(m) & & \big\downarrow m \\ & \underset{h}{\searrow} & & & \\ & & FU' & & U' \end{array} \tag{5.1}$$

That is, any map of the form $h\colon B \to FU'$ can be factored through u. Also, it intuitively means that in order to go out of U to U' in **C** it is enough to go from B to FU' in **D**. And it also means that (U, u) is an initial object in the comma category $B \downarrow F$.

Examples **5.1**

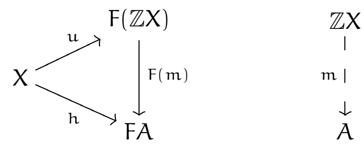
1. Consider the forgetful functor $F\colon \mathbf{Ab} \to \mathbf{Set}$ and $X \in \operatorname{obj} \mathbf{Set}$. Then a universal from X to F is the pair $(\mathbb{Z}X, u)$, where $\mathbb{Z}X$ is the free abelian group with base X and u is the inclusion $u\colon X \to F(\mathbb{Z}X)$. Clearly, if A is an abelian group, and if

$$h\colon X \to FA \tag{5.2}$$

is a map of sets, then there is a unique map of abelian groups

$$m\colon \mathbb{Z}X \to A \tag{5.3}$$

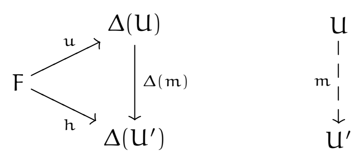
that makes the following diagram commute:



(5.4)

That is, in order to define a homomorphism from $\mathbb{Z}X$ to the abelian group A, it is enough to define a set map from X to the set FA, the underlying set of the group A.

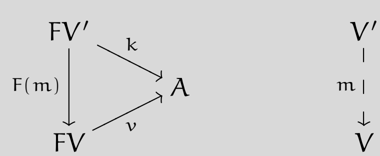
2. Let $\mathbf{I}$ be an small category, $\mathbf{C}$ an arbitrary category, and $\Delta\colon \mathbf{C} \to \mathbf{C}^{\mathbf{I}}$ the diagonal functor. If $F \in \operatorname{obj} \mathbf{C}^{\mathbf{I}}$, a $\mathbf{C}^{\mathbf{I}}$-map $u\colon F \to \Delta(U)$, with $U \in \operatorname{obj} \mathbf{C}$ is the same as a natural sink from F to U. We have that (U, u) is a universal from F to Δ precisely when the sink is limiting. The diagram then looks like:



(5.5)

With the setup of Definition 5.1, if (U, u) and (U', u') are universals from B to F, then there is a unique $\mathbf{C}$-isomorphism $m\colon U \to U'$ such that $u' = F(h) \circ u$.

5.2 Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, $F\colon \mathbf{C} \to \mathbf{D}$ be a functor, and $A \in \operatorname{obj} \mathbf{D}$. Then a *universal* from F to A is a pair (V, v) where $V \in \operatorname{obj} \mathbf{C}$ and $v\colon FV \to A$ is a $\mathbf{D}$-map, such that if $k\colon FV' \to A$ is any $\mathbf{D}$-map with $V' \in \operatorname{obj} \mathbf{C}$, then there is a unique $\mathbf{C}$-map $m\colon V' \to V$ such that the following diagram commutes:



(5.6)

5.2 Adjoint Functors

5.3 Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and $T\colon \mathbf{C} \to \mathbf{D}$, $S\colon \mathbf{D} \to \mathbf{C}$ be functors. We say that S is left adjoint to T if for all $A \in \operatorname{obj}\mathbf{D}$, $B \in \operatorname{obj}\mathbf{C}$ we have a bijective map $\eta_{A,B}$ from $\hom_{\mathbf{C}}(SA, B)$ to $\hom_{\mathbf{D}}(A, TB)$ which is natural in A and B. We denote this as $S \dashv T$, and the map η is called the *adjugant* of the adjunction $S \dashv T$.

In other words, this means that for any $A \in \operatorname{obj}\mathbf{D}$, the functors $\hom_{\mathbf{C}}(SA, -)$ and $\hom_{\mathbf{D}}(A, T-)$ are naturally isomorphic, and that for any $B \in \operatorname{obj}\mathbf{C}$, the contravariant functors $\hom_{\mathbf{C}}(S-, B)$ and $\hom_{\mathbf{D}}(-, TB)$ are naturally isomorphic, that is, the following diagram commutes for all $A \in \operatorname{obj}\mathbf{D}$ and all maps $f\colon B \to B'$ in $\mathbf{C}$

$$\begin{array}{cccc} B & \hom_{\mathbf{C}}(SA, B) & \xrightarrow{\eta_{A,B}} & \hom_{\mathbf{D}}(A, TB) \\ f\big\downarrow & f_*\big\downarrow & & \big\downarrow(Tf)_* \\ B' & \hom_{\mathbf{C}}(SA, B') & \xrightarrow{\eta_{A,B'}} & \hom(A, TB') \end{array} \tag{5.7}$$

and the following diagram commutes for all $B \in \operatorname{obj}\mathbf{C}$ and all maps $g\colon A \to A'$ in $\mathbf{D}$

$$\begin{array}{cccc} A & \hom_{\mathbf{C}}(SA, B) & \xrightarrow{\eta_{A,B}} & \hom(A, TB) \\ g\big\downarrow & (Sg)^*\big\uparrow & & \big\uparrow g^* \\ A' & \hom(SA', B) & \xrightarrow{\eta_{A',B}} & \hom(A', TB) \end{array} \tag{5.8}$$

Example 5.2

Let $\mathbf{C}$ be a cocomplete category. Then for any small category $\mathbf{I}$ we have that $\operatorname{colim}\colon \mathbf{C}^{\mathbf{I}} \to \mathbf{C}$ is left adjoint to the diagonal functor $\Delta\colon \mathbf{C} \to \mathbf{C}^{\mathbf{I}}$. And if $\mathbf{C}$ is a complete category, we have that $\lim\colon \mathbf{C}^{\mathbf{I}} \to \mathbf{C}$ is right adjoint to the diagonal.

Now, as in the situation of Definition 5.3, suppose that S and T are functors such that $S \dashv T$. For each object A of $\mathbf{D}$, let $\varepsilon_A\colon A \to TSA$ be $\varepsilon_A = \eta_{A,SA}(1_{SA})$. It can be proven that the collection of maps $\varepsilon = \{\varepsilon_A\}$ gives a natural transformation $1 \to TS$. Similarly, for each $B \in \operatorname{obj}\mathbf{C}$, let $\chi_B = \eta^{-1}_{TB,B}(1_{TB})\colon STB \to B$, then $\chi = \{\chi_A\}$ can be shown to be a natural transformation $ST \to 1$.

5.4 Definition. If S and T are functors such that $S \dashv T$, the natural transformation ε described in the previous paragraph is called the *unit* of the adjunction $S \dashv T$. The natural transformation χ is called the *counit*.

5.5 Proposition. *If S and T are functors such that $S \dashv T$ with unit $\varepsilon\colon 1 \to TS$ and counit $\chi\colon ST \to 1$, then the compositions:*

$$S \xrightarrow{S\varepsilon} STS \xrightarrow{\chi S} S, \tag{5.9}$$

$$T \xrightarrow{\varepsilon T} TST \xrightarrow{T\chi} T \tag{5.10}$$

are equal to the respective identities. Conversely, let $S\colon \mathbf{D} \to \mathbf{C}$ and $T\colon \mathbf{C} \to \mathbf{D}$ be functors such that there are natural transformations $\varepsilon\colon 1 \to TS$, $\chi\colon ST \to 1$ such that the compositions 5.9 are equal to the identity. Then $S \dashv T$, and ε is the unit of the adjunction and χ the counit. The adjugant $\eta_{A,B}$ can be obtained as follows: $\eta_{A,B}(h)$ is the composition

$$B \xrightarrow{\varepsilon_B} TSB \xrightarrow{Th} TA \tag{5.11}$$

5.6 Theorem. *Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and $T\colon \mathbf{C} \to \mathbf{D}$, $S\colon \mathbf{D} \to \mathbf{C}$ be functors such that $S \dashv T$. Then*

1. *S preserves all colimits.*
2. *T preserves all limits.*

Proof. We show that T preserves all limits. Let $F\colon \mathbf{I} \to \mathbf{C}$ be a functor with limiting source $\{\delta_i\colon X \to F_i\}$, we need to show that $\{T\delta_i\colon TX \to TF_i\}$ is a limiting source. Let $\{\rho_i\colon Z \to TF_i\}$ be a natural source, we want to show there is a unique $\mathbf{D}$-morphism $M\colon Z \to TX$ that makes the following diagram commute

$$\begin{array}{ccc} Z & \overset{M}{\dashrightarrow} & TX \\ & \underset{\rho_i}{\searrow} \quad \underset{T\delta_i}{\swarrow} & \\ & TF_i & \end{array} \tag{5.12}$$

Consider the adjunction map $\eta_{Z,F_i}\colon \hom_{\mathbf{C}}(SZ, F_i) \to \hom_{\mathbf{D}}(Z, TF_i)$. We get maps $\eta^{-1}_{Z,F_i}(\rho_i)\colon SZ \to F_i$, we want to show they form a natural source. Let $m\colon i \to j$ be an $\mathbf{I}$-morphism, we want to prove that the diagram

$$\begin{array}{ccc} & SZ & \\ & {\scriptstyle \eta^{-1}_{Z,F_i}(\rho_i)} \swarrow \quad \searrow {\scriptstyle \eta^{-1}_{Z,F_i}(\rho_j)} & \\ F_i & \xrightarrow{F(m)} & F_j \end{array} \tag{5.13}$$

is commutative. But this follows from the commutativity of the diagram

$$\begin{array}{ccccc} i & F_i & \hom_{\mathbf{C}}(SZ, F_i) & \xrightarrow{\eta_{Z,F_i}} & \hom_{\mathbf{D}}(Z, TF_i) \\ {\scriptstyle m}\downarrow & {\scriptstyle F(m)}\downarrow & {\scriptstyle F(m)_*}\downarrow & & \downarrow{\scriptstyle TF(m)_*} \\ j & F_j & \hom_{\mathbf{C}}(SZ, F_j) & \xrightarrow{\eta_{Z,F_j}} & \hom_{\mathbf{D}}(Z, TF_j) \end{array} \tag{5.14}$$

since $\rho_j = TF(m) \circ \rho_i$. We obtain then that there is a unique map $M' \colon SZ \to X$ making the diagram

$$\begin{array}{ccc} SZ & \overset{M'}{\dashrightarrow} & X \\ & {\scriptstyle \eta^{-1}_{Z,F_i}(\rho_i)} \searrow \quad \swarrow {\scriptstyle \delta_i} & \\ & F_i & \end{array} \tag{5.15}$$

commute for all $i \in \operatorname{obj} \mathbf{I}$. From the adjunction map $\eta_{Z,X} \colon \hom_{\mathbf{C}}(SZ, X) \to \hom_{\mathbf{D}}(Z, TX)$, let $M = \eta_{Z,X}(M')$, we want now to prove that the diagram (5.12) commutes for each $i \in \operatorname{obj} \mathbf{I}$. This follows from commutativity of the diagram

$$\begin{array}{cccc} X & \hom_{\mathbf{C}}(SZ, X) & \xrightarrow{\eta_{Z,X}} & \hom_{\mathbf{D}}(Z, TX) \\ {\scriptstyle \delta_i}\downarrow & {\scriptstyle (\delta_i)_*}\downarrow & & \downarrow{\scriptstyle (T\delta_i)_*} \\ F_i & \hom_{\mathbf{C}}(SZ, F_i) & \xrightarrow{\eta_{Z,F_i}} & \hom_{\mathbf{D}}(Z, TF_i) \end{array} \tag{5.16}$$

and evaluating at M' in the upper left corner. □

6 More on Limits

6.1 Limits in a Functor Category

6.1 *Theorem.* *Let* $\mathbf{I}$, $\mathbf{C}$ *be small categories,* $\mathbf{D}$ *be a cocomplete category, and* $T\colon \mathbf{I} \to \mathbf{D}^{\mathbf{C}}$. *Then the functor* T *has a colimit, which can be calculated point-wise.*

Proof. Step 1. We prove that for each $C \in \operatorname{obj} \mathbf{C}$, there is a functor $\mathbf{I} \to \mathbf{D}$ defined in objects by $i \mapsto T(i)(C)$, which we will denote as $T(-)(C)$: Let $m\colon i \to j$ be a map in $\mathbf{I}$, we need to define a $\mathbf{D}$-map $T(m)(C)\colon T(i)(C) \to T(j)(C)$. Then $T(m)\colon T(i) \to T(j)$ is a natural transformation of functors $\mathbf{C} \to \mathbf{D}$, we define $T(m)(C)$ as the C-component of the natural transformation $T(m)$. Using the definition of natural transformation, it is clear that it preserves compositions and so we have defined a functor $T(-)(C)$.

Step 2. We show there is a functor $\mathbf{C} \to \mathbf{D}$ defined in objects as $C \to \operatorname{colim}_l T(l)(C)$, which we will denote as $\operatorname{colim}_l T(l)(-)$: Let $f\colon C \to C'$ be a $\mathbf{C}$-morphism, we want to define a natural transformation $T(-)(f)\colon T(-)(C) \to T(-)(C')\colon \mathbf{I} \to \mathbf{D}$. The i-component of that is precisely $T(i)(f)$. This is indeed a natural transformation, since given an $\mathbf{I}$-map $m\colon i \to j$, we get the following commutative diagram, using that $T(m)\colon T(i) \to T(j)$ is a natural transformation:

$$\begin{array}{ccccc} C & \quad & T(i)(C) & \xrightarrow{T(m)(C)} & T(j)(C) \\ {\scriptstyle f}\downarrow & & {\scriptstyle T(i)(f)}\downarrow & & \downarrow{\scriptstyle T(j)(f)} \\ C' & & T(i)(C') & \xrightarrow{T(m)(C')} & T(j)(C') \end{array} \tag{6.1}$$

which we can interpret as:

$$\begin{array}{ccccc} i & & T(i)(C) & \xrightarrow{T(i)(f)} & T(i)(C') \\ {\scriptstyle m}\downarrow & & {\scriptstyle T(m)(C)}\downarrow & & \downarrow{\scriptstyle T(m)(C')} \\ j & & T(j)(C') & \xrightarrow{T(j)(f)} & T(j)(C') \end{array} \tag{6.2}$$

Hence the natural transformation $T(-)(f)\colon T(-)(C) \to T(-)(C')$ induces a **D**-morphism $\operatorname{colim}_l T(l)(C) \to \operatorname{colim}_l T(l)(C')$, which we will denote as $\operatorname{colim}_l T(l)(f)$.

For the record, we note that this morphism has the property of being the unique **D**-map that makes the following diagram commute for all $i \in \operatorname{obj} \mathbf{I}$.

$$\begin{array}{ccc} T(i)(C) & \xrightarrow{T(i)(f)} & T(i)(C') \\ {\scriptstyle \varepsilon_i^C}\downarrow & & \downarrow{\scriptstyle \varepsilon_i^{C'}} \\ \operatorname{colim}_l T(l)(C) & \xrightarrow{\operatorname{colim}_l T(l)(f)} & \operatorname{colim}_l T(l)(C') \end{array} \tag{6.3}$$

Step 3. We construct a sink ε_i from $T\colon \mathbf{I} \to \mathbf{D}^{\mathbf{C}}$ to the $\mathbf{D}^{\mathbf{C}}$-object $\operatorname{colim}_l T(l)(-)$: We define a collection of $\mathbf{D}^{\mathbf{C}}$-morphisms $T(i) \to \operatorname{colim}_l T(l)(-)$, one for each $i \in \operatorname{obj} \mathbf{I}$. For $C \in \operatorname{obj} \mathbf{C}$, we set the C-component of ε_i as the map ε_i^C of diagram (6.3). This is a natural transformation, since the diagram:

$$\begin{array}{ccccc} C & & T(i)(C) & \xrightarrow{\varepsilon_i^C} & \operatorname{colim}_l T(l)(C) \\ {\scriptstyle f}\downarrow & & {\scriptstyle T(i)(f)}\downarrow & & \downarrow{\scriptstyle \operatorname{colim}_l T(l)(f)} \\ C' & & T(i)(C') & \xrightarrow{\varepsilon_i^{C'}} & \operatorname{colim}_l T(l)(C') \end{array} \tag{6.4}$$

is diagram (6.3). So $\{\varepsilon_i\}$ is a sink.

Step 4. We show that $\{\varepsilon_i\}$ is a natural sink: Let $m\colon i \to j$ be an **I**-morphism, we need to show commutativity of

$$\begin{array}{ccc} T(i) & \xrightarrow{T(m)} & T(j) \\ & {\scriptstyle \varepsilon_i}\searrow \quad \swarrow{\scriptstyle \varepsilon_j} & \\ & \operatorname{colim}_l T(l)(-) & \end{array} \tag{6.5}$$

which follows from commutativity of

$$\begin{array}{ccc} T(i)(C) & \xrightarrow{T(m)(C)} & T(j)(C) \\ & {\scriptstyle \varepsilon_i^C}\searrow \quad \swarrow{\scriptstyle \varepsilon_j^C} & \\ & \operatorname{colim}_l T(l)(C) & \end{array} \tag{6.6}$$

for each $C \in \operatorname{obj} \mathbf{C}$, since this is precisely the condition that defines the maps ε_i^C.

Step 5. We show that the maps $\varepsilon_i \colon T(i) \to \operatorname{colim}_\iota T(\iota)(-)$ form a limiting sink, that is, the functor $\operatorname{colim}_\iota T(\iota)(-) \colon \mathbf{C} \to \mathbf{D}$ is the colimit of $T \colon \mathbf{I} \to \mathbf{D}^{\mathbf{C}}$. Let $\varepsilon_i' \colon T(i) \to Z$ be a natural sink. Evaluating at $C \in \operatorname{obj} \mathbf{C}$, we get a natural sink $(\varepsilon_i')^C \colon T(i)(C) \to Z(C)$, and so we obtain a map M_C making the following diagram commute:

$$\begin{array}{ccc} & T(i)(C) & \\ {\scriptstyle \varepsilon_i^C}\swarrow & & \searrow{\scriptstyle (\varepsilon_i')^C} \\ \operatorname{colim}_\iota T(\iota)(C) & \xrightarrow{M_C} & Z(C) \end{array} \tag{6.7}$$

We want to show that the M_C are the components of a natural transformation $\operatorname{colim}_\iota T(\iota)(-) \to Z$, that is, that the following diagram is commutative for all $\mathbf{C}$-maps $f \colon C \to C'$.

$$\begin{array}{cccc} C & \operatorname{colim}_\iota T(\iota)(C) & \xrightarrow{M_C} & Z(C) \\ {\scriptstyle f}\downarrow & {\scriptstyle \operatorname{colim}_\iota T(\iota)(f)}\downarrow & & \downarrow{\scriptstyle Z(f)} \\ C' & \operatorname{colim}_\iota T(\iota)(C') & \xrightarrow{M_{C'}} & Z(C') \end{array} \tag{6.8}$$

Since $\varepsilon_i' \colon T(i) \to Z$ is a natural transformation, we have commutativity of

$$\begin{array}{cccc} C & T(i)(C) & \xrightarrow{(\varepsilon_i')^C} & Z(C) \\ {\scriptstyle f}\downarrow & {\scriptstyle T(i)(f)}\downarrow & & \downarrow{\scriptstyle Z(f)} \\ C' & T(i)(C') & \xrightarrow{(\varepsilon_i')^{C'}} & Z(C') \end{array} \tag{6.9}$$

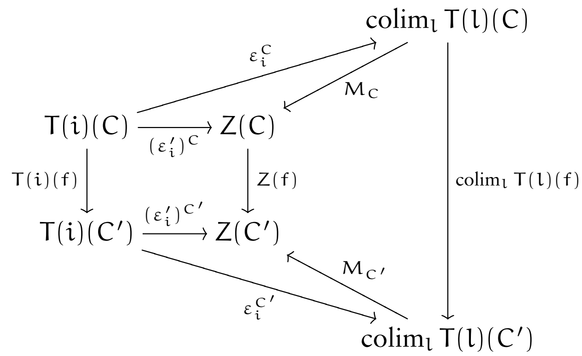
which combined with diagrams (6.3) and (6.7) results in the diagram of Figure 6.1.

From there, we get that both $Z(f) \circ M_C$ and $M_{C'} \circ \operatorname{colim}_\iota T(\iota)(f)$, when in the place of the dotted arrow, make the following diagram commute for all $C \in \operatorname{obj} \mathbf{C}$:

$$\begin{array}{ccc} & T(i)(C) & \\ {\scriptstyle \varepsilon_i^C}\swarrow & & \searrow{\scriptstyle Z(f)\circ(\varepsilon_i')^C} \\ \operatorname{colim}_\iota T(\iota)(C) & \dashrightarrow & Z(C') \end{array} \tag{6.11}$$

By uniqueness, given that the ε_i^C form a limiting sink, we have that $Z(f) \circ M_C = M_{C'} \circ \operatorname{colim}_\iota T(\iota)(f)$, and hence we obtain the result. □

6.2 *Corollary.* *Suppose we have a functor* $S \colon \mathbf{C} \times \mathbf{I} \to \mathbf{D}$. *We can interpret* S *as a functor with parameter* i.



(6.10)

Figure 6.1: Diagram

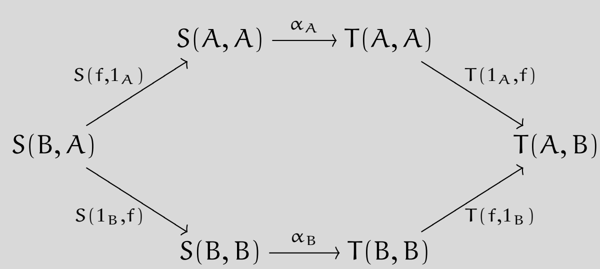
6.3 *Corollary.*

If $\mathbf{D}$ *is a complete category,* $\mathbf{C}$ *is a small category and* $F_1, \dots, F_n$ *are functors* $\mathbf{C} \to \mathbf{D}$*, then we have a product functor,* $\prod_{i=1}^{n} F_j \colon \mathbf{C} \to \mathbf{D}$*, defined as* $(\prod F_j)(A) = \prod F_j(A)$*, for* $A \in \operatorname{obj} \mathbf{C}$*.*

6.2 Ends

6.4 Definition.

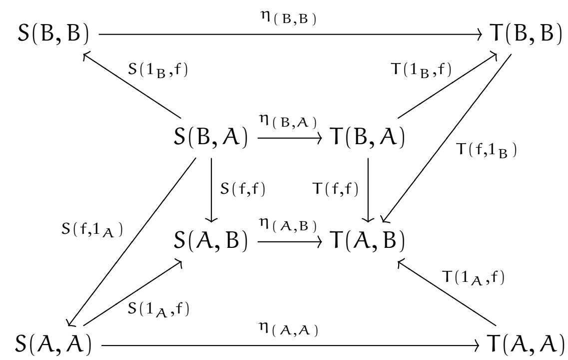
Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and S, T be functors $\mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$. A *dinatural transformation* $\alpha \colon S \to T$ is a collection of $\mathbf{D}$-maps, $\alpha_A \colon S(A, A) \to T(A, A)$, indexed by the objects of $\mathbf{C}$, such that the following diagram commutes for every $\mathbf{C}$-map $f \colon A \to B$.



(6.12)

Example **6.1**

If $\eta\colon S \to T$ is a natural transformation, then $\{\eta_{(C,C)}\}$ is a dinatural transformation. Consider the following diagram:

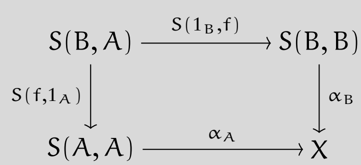


(6.13)

Then we have:

$$T(1_A, f) \circ \eta_{(A,A)} \circ S(f, 1_A) = \eta_{(A,B)} \circ S(1_A, f) \circ S(f, 1_A) \tag{6.14}$$
$$= \eta_{(A,B)} \circ S(f, f) \tag{6.15}$$
$$= T(f, f) \circ \eta_{(B,A)} \tag{6.16}$$
$$= T(f, 1_B) \circ T(1_B, f) \circ \eta_{(B,A)} \tag{6.17}$$
$$= T(f, 1_B) \circ \eta_{(B,B)} \circ S(1_B, f) \tag{6.18}$$

6.5 Definition. A *dinatural sink* α from a functor $S\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$ to $X \in \operatorname{obj} \mathbf{D}$ is a dinatural transformation from the functor S to the constant functor $\mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$ with value X. In detail, it is a collection of **D**-maps $\alpha_C\colon S(C, C) \to X$, indexed by the objects of **C**, such that for every $f\colon A \to B$ the following diagram



(6.19)

commutes.

Example 6.2

Let $X \in \operatorname{obj} \mathbf{Set}$ fixed. For each $A \in \operatorname{obj} \mathbf{Set}$, we have a set map $e_A\colon \hom_{\mathbf{Set}}(A, X) \times A \to X$, given by evaluation, that is, $e_A(f, a) = f(a)$. The maps $\{e_A\}$ form a dinatural sink from the functor $\hom_{\mathbf{Set}}(-, X) \times (-)\colon \mathbf{Set}^{op} \times \mathbf{Set} \to \mathbf{Set}$ to the set X, since for every set map $\phi\colon A \to B$

the following square commutes:

$$\begin{array}{ccc} \hom_{\mathbf{Set}}(B,X)\times A & \xrightarrow{1\times\phi} & \hom_{\mathbf{Set}}(B,X)\times B \\ {\scriptstyle \phi^*\times 1_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ \hom_{\mathbf{Set}}(A,X)\times A & \xrightarrow{\alpha_A} & X \end{array} \tag{6.20}$$

6.6 Definition. A *dinatural source* β from $Y \in \operatorname{obj}\mathbf{D}$ to the functor $S\colon \mathbf{C}^{\mathrm{op}}\times\mathbf{C}\to\mathbf{D}$ it is a collection of $\mathbf{D}$-maps $\{\beta_C\colon Y\to S(C,C)\}_{C\in\operatorname{obj}\mathbf{C}}$ such that for every $f\colon A\to B$ the following diagram commutes

$$\begin{array}{ccc} Y & \xrightarrow{\beta_B} & S(B,B) \\ {\scriptstyle \beta_A}\downarrow & & \downarrow{\scriptstyle S(f,1_B)} \\ S(A,A) & \xrightarrow{S(1_A,f)} & S(A,B) \end{array} \tag{6.21}$$

6.7 Definition. An *end* of a functor $S\colon \mathbf{C}^{\mathrm{op}}\times\mathbf{C}\to\mathbf{D}$ is a dinatural source from an object Y to S, such that for every dinatural source β' from S to some object Y' there is a unique $\mathbf{D}$-map $Y'\to Y$ such that the following diagram commutes.

$$\begin{array}{ccc} Y' & \overset{M}{\dashrightarrow} & Y \\ & {\scriptstyle \beta'_C}\searrow \quad \swarrow{\scriptstyle \beta_C} & \\ & S(C,C) & \end{array} \tag{6.22}$$

for every $C\in\operatorname{obj}\mathbf{C}$. We denote this as

$$Y = \int_C S(C,C) \tag{6.23}$$

and we also say that the dinatural source β is *ending*.

Example 6.3

Let $\mathbf{C}$ be a small category, and $F, F'\colon \mathbf{C}\to\mathbf{D}$ be two functors. We can then consider the functor $\mathbf{C}^{\mathrm{op}}\times\mathbf{C}\to\mathbf{Set}$ given by $(A,B)\mapsto \hom_{\mathbf{D}}(FA,F'B)$. We claim that $\int_A \hom_{\mathbf{D}}(FA,F'A) = \hom_{\mathbf{D^C}}(F,F')$, the set of natural transformations from F to F'. Let $\beta_C\colon \hom_{\mathbf{D^C}}(F,F')\to \hom_{\mathbf{D}}(FC,F'C)$ be given by $\beta_C(\eta)=\eta_C$. Then β is dinatural, since the condition of square (6.21) says in this case

$$\begin{array}{ccc} \hom_{\mathbf{D^C}}(F,F') & \xrightarrow{\beta_B} & \hom_{\mathbf{D}}(FB,F'B) \\ {\scriptstyle\beta_A}\downarrow & & \downarrow{\scriptstyle(Ff)^*} \\ \hom_{\mathbf{D}}(FA,F'A) & \xrightarrow{(F'f)_*} & \hom_{\mathbf{D}}(FA,F'B) \end{array} \tag{6.24}$$

which holds, since for $\eta \in \hom_{\mathbf{D^C}}(F,F')$, we have that $\eta_B \circ F(f) = F'(f) \circ \eta_A$. If $\beta'_C\colon Z \to \hom_{\mathbf{D}}(FC,F'C)$ is another dinatural source with Z a set, then for each $z \in Z$ one gets a $\mathbf{D}$-map $\beta'_C(z)\colon FC \to F'C$, and the collection $\{\beta'_C(z)\}_{C\in \operatorname{obj}\mathbf{C}}$ is a natural transformation $F \to F'$. Hence we have a map $M\colon Z \to \hom_{\mathbf{D^C}}(F,F')$ given by $z \mapsto \{\beta'_C(z)\}_{C\in\operatorname{obj}\mathbf{C}}$, which satisfies the commutativity condition 6.22.

6.8 *Theorem.* *Let* $\gamma\colon S \to S'$ *be a natural transformation between functors* $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ *with ends. Then there is a unique* $\mathbf{D}$*-map* $\int_C \gamma_{C,C}\colon \int_C S \to \int_C S'$ *such that the following diagram commutes for every* $C \in \operatorname{obj}\mathbf{C}$.

$$\begin{array}{ccc} \int_C S(C,C) & \overset{\int_C \gamma}{\dashrightarrow} & \int_C S'(C,C) \\ {\scriptstyle\beta_C}\downarrow & & \downarrow{\scriptstyle\beta'_C} \\ S(C,C) & \xrightarrow{\gamma_{C,C}} & S'(C,C) \end{array} \tag{6.25}$$

Proof. The collection of compositions $\{\int_C S \xrightarrow{\beta_C} S(C,C) \xrightarrow{\gamma_{C,C}} S'(C,C)\}_{C\in\operatorname{obj}\mathbf{C}}$, is a dinatural source, because of the diagram:

$$\begin{array}{ccccc} \int S & \xrightarrow{\beta_B} & S(B,B) & \xrightarrow{\gamma_{B,B}} & S'(B,B) \\ {\scriptstyle\beta_A}\downarrow & & \downarrow{\scriptstyle S(f,1_B)} & & \\ S(A,A) & \xrightarrow{S(1_A,f)} & S(A,B) & & \downarrow{\scriptstyle S'(f,1_B)} \\ {\scriptstyle\gamma_{A,A}}\downarrow & & & \searrow^{\gamma_{A,B}} & \\ S'(A,A) & & \xrightarrow{S(1_A,f)} & & S'(A,B) \end{array} \tag{6.26}$$

where the two trapezoids commute because of the naturality of γ. By definition of $\int S'$, there is a unique map $M\colon \int S \to \int S'$ making the following triangle commute for all

$C \in \operatorname{obj} \mathbf{C}$,

$$\begin{array}{ccc} \int S & \overset{M}{\dashrightarrow} & \int S' \\ & {\scriptstyle \gamma \circ \beta_C} \searrow \quad \swarrow {\scriptstyle \beta'_C} & \\ & S'(C, C) & \end{array} \tag{6.27}$$

then let $M = \int g$. □

✎ Exercises 6.1

1. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor, and $S\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$ be the functor that is the composition of projection onto the second factor from $\mathbf{C}^{op} \times \mathbf{C}$, with F (that is, $S(A, B) = F(B)$). Then $\int_C S(C, C) = \lim F$.
2. Let $S\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$ be a functor, $\{\beta_C\colon Y \to S(C, C)\}_{C \in \operatorname{obj} \mathbf{C}}$ be a dinatural source, and $\phi\colon Y' \to Y$ be a $\mathbf{C}$-map. Then $\{\beta_C \circ \phi\colon Y' \to S(C, C)\}_{C \in \operatorname{obj} \mathbf{C}}$ is a dinatural source.
3. Let $S\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$ be a functor, $\{\beta_C\colon Y \to S(C, C)\}_{C \in \operatorname{obj} \mathbf{C}}$ be an ending source, and $\phi, \psi\colon Y' \to Y$ $\mathbf{C}$-maps such that $\beta_C \circ \phi = \beta_C \circ \psi$ for all $C \in \operatorname{obj} \mathbf{C}$. Then $\phi = \psi$.

6.3 Ends in a Functor Category

6.9 ***Theorem.*** *Let $\mathbf{C}$, $\mathbf{E}$ be small categories, $\mathbf{D}$ complete and $T\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}^{\mathbf{E}}$ be a functor. Then T has an end, which can be calculated pointwise.*

Proof. Step 1. We show that for each $E \in \operatorname{obj} \mathbf{E}$, there is a functor $\mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$, defined on objects as $(A, B) \mapsto T(A, B)(E)$. Let $(\phi, \psi)\colon (A, B) \to (A', B')$ be a map in $\mathbf{C}^{op} \times \mathbf{C}$. Then $T(\phi, \psi)\colon T(A, B) \to T(A', B')$ is a natural transformation, so we define $T(\phi, \psi)(E)$ as the E-component of it.

Step 2. We show that there is a functor $\mathbf{E} \to \mathbf{D}$, defined on objects as $E \mapsto \int_C T(C, C)(E)$, which we will denote as $\int_C T(C, C)(E)$: First, for an $\mathbf{E}$-morphism $f\colon E \to E'$, we want a natural transformation $T(-,-)(E) \to T(-,-)(E')$, that is, for each $(A, B) \in \operatorname{obj} \mathbf{C}^{op} \times \mathbf{C}$, we need a map $T(A,B)(E) \to T(A,B)(E')$. Let it be $T(A,B)(f)$. We need to show it is natural, that is

$$\begin{array}{ccc} (A,B) & & T(A,B)(E) \xrightarrow{T(A,B)(f)} T(A,B)(E') \\ {\scriptstyle (\phi,\psi)}\downarrow & & {\scriptstyle T(\phi,\psi)(E')}\downarrow \qquad\qquad\qquad \downarrow{\scriptstyle T(\phi,\psi)(E)} \\ (A',B') & & T(A',B')(E) \xrightarrow{T(A',B')(f)} T(A',B')(E') \end{array} \tag{6.28}$$

but this follows from the diagram

$$\begin{array}{ccc} E & & T(A,B)(E) \xrightarrow{T(\phi,\psi)(E)} T(A',B')(E) \\ {\scriptstyle f}\downarrow & & {\scriptstyle T(A,B)(f)}\downarrow \qquad\qquad\qquad \downarrow{\scriptstyle T(A',B')(f)} \\ E' & & T(A,B)(E') \xrightarrow{T(\phi,\psi)(E')} T(A',B')(E') \end{array} \tag{6.29}$$

We note that, by Theorem 6.8, the natural transformation $T(-,-)(f)$ induces a **D**-map $\int_C T(C,C)(f)\colon \int_C T(C,C)(E) \to \int_C T(C,C)(E')$, which has the property of being the unique map that makes the following diagram commute for all $A \in \operatorname{obj} \mathbf{C}$.

$$\begin{array}{ccc} \int_C T(C,C)(E) & \xrightarrow{\int_C T(C,C)(f)} & \int_C T(C,C)(E') \\ {\scriptstyle \beta_A^E}\downarrow & & \downarrow{\scriptstyle \beta_A^{E'}} \\ T(A,A)(E) & \xrightarrow{T(A,A)(f)} & T(A,A)(E') \end{array} \tag{6.30}$$

Step 3. We construct a dinatural source from the $\mathbf{D}^{\mathbf{E}}$-object $\int_C T(C,C)(-)$ to T. We need to define a collection of $\mathbf{D}^{\mathbf{E}}$-morphisms $\beta_A\colon \int_C T(C,C)(-) \to T(A,A)$, one for each $A \in \operatorname{obj} \mathbf{C}$. We set β_A^E as in diagram (6.30). That $\{\beta_A^E\}_{A\in \operatorname{obj} C}$ is a natural transformation follows from that diagram. We now show that it is dinatural, that is, that for every **C**-map $f\colon A \to B$ the following square commutes:

$$\begin{array}{ccc} \int_C T(C,C)(-) & \xrightarrow{\beta_B} & T(B,B) \\ {\scriptstyle \beta_A}\downarrow & & \downarrow{\scriptstyle T(f,1_B)} \\ T(A,A) & \xrightarrow{T(1_A,f)} & T(A,B) \end{array} \tag{6.31}$$

Evaluating at E, we get

$$\begin{array}{ccc} \int_C T(C,C)(E) & \xrightarrow{\beta_B^E} & T(B,B)(E) \\ {\scriptstyle \beta_A^E}\downarrow & & \downarrow{\scriptstyle T(f,1_B)(E)} \\ T(A,A)(E) & \xrightarrow{T(1_A,f)(E)} & T(A,B)(E) \end{array} \tag{6.32}$$

But this is precisely the condition that defines $\int_C T(C,C)(E)$, and so the diagram (6.31) commutes.

Step 4. This dinatural source is an ending source. Let $\beta'_A \colon Z \to T(A,A)$ be a dinatural source, with $Z \in \operatorname{obj} \mathbf{D}^{\mathbf{E}}$. Evaluating at E, we obtain a map M_E that makes the following diagram commute:

$$\begin{array}{ccc} Z(E) & \overset{M_E}{\dashrightarrow} & \int_C T(C,C)(E) \\ & {\scriptstyle (\beta'_A)^E} \searrow \quad \swarrow {\scriptstyle \beta^E_A} & \\ & T(A,A)(E) & \end{array} \tag{6.33}$$

We want to show that the collection of maps $\{M_E\}_{E \in \operatorname{obj} \mathbf{E}}$ is a natural transformation $Z \to \int_C T(C,C)(-)$, that is,

$$\begin{array}{c} E \\ {\scriptstyle f}\downarrow \\ E' \end{array} \qquad \begin{array}{ccc} Z(E) & \xrightarrow{M_E} & \int_C T(C,C)(E) \\ {\scriptstyle Z(f)}\downarrow & & \downarrow{\scriptstyle \int_C T(C,C)(f)} \\ Z(E') & \xrightarrow{M_{E'}} & \int_C T(C,C)(E') \end{array} \tag{6.34}$$

Since $\beta'_A \colon Z \to T(A,A)$ is a natural transformation, we have commutativity of the small rectangle in the following diagram:

$$\begin{array}{cccc} \int_C T(C,C)(E) & & \xrightarrow{\beta^E_A} & \\ & \overset{M_E}{\nwarrow} \; Z(E) & \xrightarrow{(\beta'_A)^E} & T(A,A)(E) \\ {\scriptstyle \int_C T(C,C)(f)}\downarrow & {\scriptstyle Z(f)}\downarrow & & \downarrow{\scriptstyle T(A,A)(f)} \\ & \underset{M_{E'}}{\swarrow} \; Z(E') & \xrightarrow{(\beta'_A)^{E'}} & T(A,A)(E') \\ \int_C T(C,C)(E') & & \xrightarrow{\beta^{E'}_A} & \end{array} \tag{6.35}$$

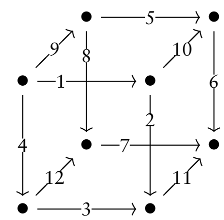
The triangles are instances of (6.33), and the biggest trapezoid is (6.30). Commutativity of the smallest trapezoid is not immediate, but it follows from the fact that both $M_{E'} \circ Z(f)$ and $\int_C T(C,C)(f) \circ M_E$ complete the following commutative triangle, when in place of the dotted arrow:

$$\begin{array}{ccc} Z(E) & \dashrightarrow & \int_C T(C,C)(E') \\ & {\scriptstyle T(A,A)(f)\circ(\beta'_A)^E} \searrow \quad \swarrow {\scriptstyle \beta^{E'}_A} & \\ & T(A,A)(E') & \end{array} \tag{6.36}$$

□

6.4 Iterated Ends

6.10 ***Lemma.*** *Consider the following diagram:*



(6.37)

Suppose that all faces of the cube are commutative squares, except maybe the front and back. Then

1. *if the front face commutes, then* $7 \circ 8 \circ 9 = 6 \circ 5 \circ 9$,
2. *if the back face commutes, then* $11 \circ 2 \circ 1 = 11 \circ 3 \circ 4$.

Proof. Exercise. □

6.11 ***Lemma.*** *Let* $\mathbf{C}$, $\mathbf{E}$ *be small categories,* $\mathbf{D}$ *an arbitrary category, and* $T\colon \mathbf{C}^{op} \times \mathbf{C} \times \mathbf{E}^{op} \times \mathbf{E} \to \mathbf{D}$ *be a functor. Let* $\kappa_{C,E}\colon Y \to T(C, C, E, E)$ *that is a dinatural source on* C *for* E *fixed, and a dinatural source on* E *for* C *fixed. Then* κ *is also a dinatural source when* T *is considered as a functor* $(\mathbf{C} \times \mathbf{E})^{op} \times (\mathbf{C} \times \mathbf{E}) \to \mathbf{D}$.

Proof. Let $f\colon A \to B$, $h\colon E \to F$ be morphisms in $\mathbf{C}$, $\mathbf{E}$ respectively. We must show commutativity of the square:

$$\begin{array}{ccc} Y & \xrightarrow{\kappa_{B,F}} & T(B,B,F,F) \\ {\scriptstyle \kappa_{A,E}}\downarrow & & \downarrow{\scriptstyle T(f,1_B,h,1_F)} \\ T(A,A,E,E) & \xrightarrow{T(1_A,f,1_E,h)} & T(A,B,E,F) \end{array} \tag{6.38}$$

Since $\kappa_{C,E}$ is dinatural in C, when the second variable is fixed we have commutativity of the square

$$\begin{array}{ccc} Y & \xrightarrow{\kappa_{B,F}} & T(B,B,F,F) \\ {\scriptstyle \kappa_{A,F}}\downarrow & & \downarrow{\scriptstyle T(f,1_B,1_F,1_F)} \\ T(A,A,F,F) & \xrightarrow{T(1_A,f,1_F,1_F)} & T(A,B,F,F) \end{array} \tag{6.39}$$

and since $\kappa_{C,E}$ is dinatural in E, when the first variable is fixed we have commutativity of the square

$$\begin{array}{ccc} Y & \xrightarrow{\kappa_{A,F}} & T(A,A,F,F) \\ {\scriptstyle \kappa_{A,E}}\downarrow & & \downarrow{\scriptstyle T(1_A,1_A,h,1_F)} \\ T(A,A,E,E) & \xrightarrow{T(1_A,1_A,1_E,h)} & T(A,A,E,F) \end{array} \tag{6.40}$$

We have then

$$\begin{aligned} T(1_A, f, 1_E, h) \circ \kappa_{A,E} &= T(1_A, f, 1_E, 1_F) \circ T(1_A, 1_A, 1_E, h) \circ \kappa_{A,E} && (6.41)\\ &= T(1_A, f, 1_E, 1_F) \circ T(1_A, 1_A, h, 1_F) \circ \kappa_{A,F} && (6.42)\\ &= T(1_A, 1_B, h, 1_F) \circ T(1_A, f, 1_F, 1_F) \circ \kappa_{A,F} && (6.43)\\ &= T(1_A, 1_B, h, 1_F) \circ T(f, 1_B, 1_F, 1_F) \circ \kappa_{B,F} && (6.44)\\ &= T(f, 1_B, h, 1_F) \circ \kappa_{B,F} && (6.45) \end{aligned}$$

□

6.12 *Theorem.* *Let* **C**, **E** *be small categories,* **D** *complete and* $T\colon \mathbf{C}^{op} \times \mathbf{C} \times \mathbf{E}^{op} \times \mathbf{E} \to \mathbf{D}$ *be a functor. Then there is an isomorphism*

$$\int_{(C,E)} T(C,C,E,E) \to \int_C \Big(\int_E T(C,C,E,E) \Big) \tag{6.46}$$

where in the first integral we have interpreted T *as a functor* $(\mathbf{C}\times\mathbf{E})^{op} \times (\mathbf{C}\times\mathbf{E}) \to \mathbf{D}$

Proof. For each $(A,B) \in \text{obj}\,\mathbf{C}^{op} \times \mathbf{C}$, we have the ending source (in E)

$$\beta_{A,B,E}\colon \int_E T(A,B,E,E) \to T(A,B,E,E), \tag{6.47}$$

and we also have the ending source in C:

$$\rho_C\colon \int_C \Big(\int_E T(C,C,E,E) \Big) \to \int_E T(C,C,E,E). \tag{6.48}$$

Hence we get a collection of maps $\kappa_{C,E}$, indexed by the objects of $\mathbf{C} \times \mathbf{E}$, given by the compositions:

$$\kappa_{C,E}\colon \int_C \Big(\int_E T(C,C,E,E) \Big) \xrightarrow{\rho_C} \int_E T(C,C,E,E) \xrightarrow{\beta_{C,C,E}} T(C,C,E,E), \tag{6.49}$$

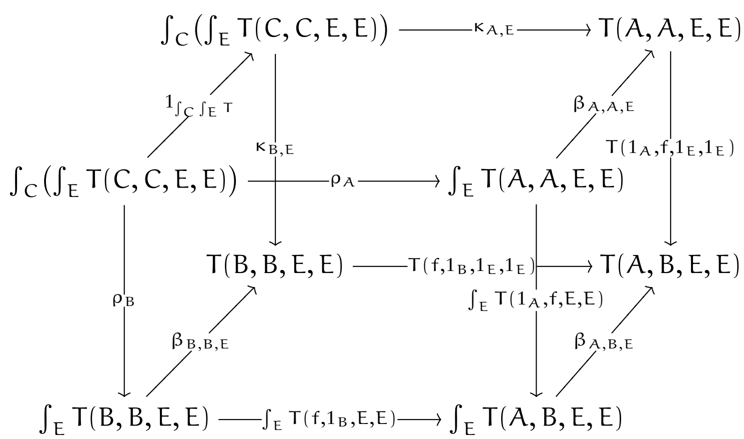
which is a dinatural source in E by Exercise 6.1.2, Let $f\colon A \to B$ be a **C**-map, and consider the square

$$\begin{CD} \int_C\left(\int_E T(C,C,E,E)\right) @>{\rho_A}>> \int_E T(A,A,E,E) \\ @V{\rho_B}VV @VV{\int_E T(1_A,f,E,E)}V \\ \int_E T(B,B,E,E) @>{\int_E T(f,1_B,E,E)}>> \int_E T(A,B,E,E) \end{CD} \tag{6.50}$$

which commutes, since $\{\rho_C\}$ is a dinatural source, and the square

$$\begin{CD} \int_C\left(\int_E T(C,C,E,E)\right) @>{\kappa_{A,E}}>> T(A,A,E,E) \\ @V{\kappa_{B,E}}VV @VV{T(1_A,f,1_E,1_E)}V \\ T(B,B,E,E) @>{T(f,1_B,1_E,1_E)}>> T(A,B,E,E) \end{CD} \tag{6.51}$$

which we want to prove commutative. Consider the cubical diagram, that has as front face the square (6.50) and as back face the square (6.51).

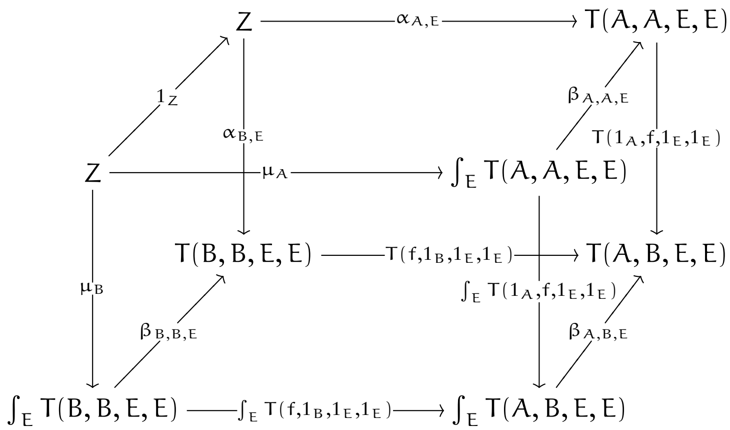


(6.52)

In this cube, the bottom and right faces commute by Theorem 6.8, and the top and left faces do by definition of κ (6.49). By Lemma 6.10, the back face commutes, and so $\kappa_{C,E}$ is a dinatural source in C. By Lemma 6.11, κ is a dinatural source in both variables. We show now that κ is a limiting source. Let $\{\alpha_{C,E}\colon Z \to T(C,C,E,E)\}$ be a dinatural source (in both variables). Since α is a dinatural source for C fixed, there is a unique map $\mu_C\colon Z \to \int_E T(C,C,E,E)$ that makes the following diagram commute

$$\begin{array}{ccc} Z & \overset{\mu_C}{\dashrightarrow} & \int_E T(C,C,E,E) \\ & {\scriptstyle\alpha_{C,E}}\searrow \quad \swarrow{\scriptstyle\beta_{C,C,E}} & \\ & T(C,C,E,E) & \end{array} \quad . \tag{6.53}$$

For $f\colon A \to B$, consider now the cubical diagram



(6.54)

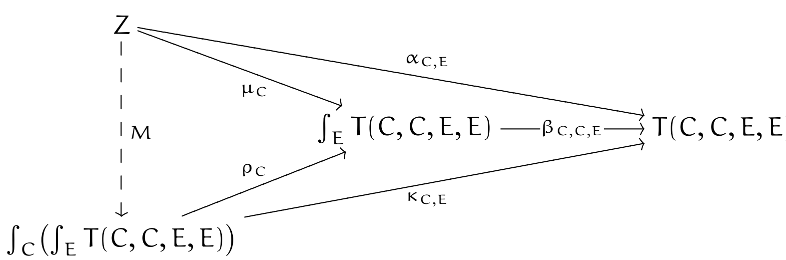
Bottom and right faces are the same of the cube (6.52), so they commute, and top and left faces commute since they are instances of (6.53). Since the back face commutes by dinaturality of α, by Lemma 6.10 we have that

$$\beta_{A,B,E} \circ \int_E T(1_A, f, 1_E, 1_E) \circ \mu_A = \beta_{A,B,E} \circ \int_E T(f, 1_B, 1_E, 1_E) \circ \mu_B \qquad (6.55)$$

for all $f\colon A \to B$ and $E \in \operatorname{obj} \mathbf{E}$. Since $\{\beta_{A,B,E}\}_{E\in\operatorname{obj}\mathbf{E}}$ is an ending source, then

$$\int_E T(1_A, f, 1_E, 1_E) \circ \mu_A = \int_E T(f, 1_B, 1_E, 1_E) \circ \mu_B \qquad (6.56)$$

by Exercise 6.1.3. Hence $\{\mu_C\}$ is a dinatural source, and so there is a unique map $M\colon Z \to \int_C\big(\int_E T(C, C, E, E)\big)$ that makes the following diagram commute



(6.57)

which finishes the proof that $\{\kappa_{C,E}\}$ is and ending source. □

✎ Exercises 6.2

1. Prove Lemma 6.10.
2. Considering again the cube (6.37) from Lemma 6.10, if all faces commute except the bottom, then $7 \circ 12 \circ 4 = 11 \circ 3 \circ 4$.

6.5 Coends

6.13 Definition. A *dinatural* source from $Y \in \operatorname{obj} \partial$ to a functor $S\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ is a dinatural transformation from the constant functor $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ with value Y to S. In other words, it is a collection of maps

Part III

Extras

7 Abelian Categories

7.1 Definition

7.1 Definition. The category $\mathbf{C}$ is an **Ab**-*category* if every set $\hom_{\mathbf{C}}(A, B)$ has a structure of abelian group in such a way that composition is distributive over the additive structure. That is, for any maps $f: A \to B$, $h_1, h_2: B \to C$ and $k: C \to D$, we have that $k \circ (h_1 + h_2) \circ f = k \circ h_1 \circ f + k \circ h_2 \circ f$.
A functor between **Ab**-categories $F: \mathbf{C} \to \mathbf{D}$ is called *additive* if for each pair of objects $A, B \in \operatorname{obj} \mathbf{C}$ we have that $F: \hom_{\mathbf{C}}(A, B) \to \hom_{\mathbf{D}}(FA, FB)$ is a group homomorphism.

Examples 7.1

1. The category **Ab** is an **Ab**-category, where we define, for $f, h \hom_{\mathbf{Ab}}(A, B)$ the sum $f + h$ as the map $x \mapsto (f + h)(x) = f(x) + h(x)$. Similarly, R-**mod** is an **Ab**-category any ring R.
2. The definition of $f + h$ as $x \mapsto (f + h)(x) = f(x) + h(x)$ does not work in the category **Rng**, since $f + h$ does not preserve in general the product in rings.
3. **Ab**-categories with only one object, can be identified with rings. If R is a ring, denote by $\mathbf{R}$ the associated **Ab**-category with one object $*$. Then an additive functor $F: \mathbf{R} \to \mathbf{Ab}$ can be identified with an structure of left R-module on the abelian group $F(*) = M$, since F maps $r \in R$ to the abelian group map $r: M \to M$, $m \mapsto rm$.

7.2 Definition. An *additive category* is an **Ab**-category with a zero object, and a product for any pair of objects.

In an additive category, the zero morphism in $\hom(A, B)$ is precisely the zero element.

7.3 Definition. An *abelian category* $\mathbf{C}$ is an additive category such that

1. Every morphism in $\mathbf{C}$ has a kernel and a cokernel.
2. Every monic in $\mathbf{C}$ is the kernel of its cokernel.
3. Every epic in $\mathbf{C}$ is the cokernel of its kernel.

[1] [2]

In an abelian category, monics and epics are usually called respectively *monomorphisms* and *epimorphisms*.

[3] [4]

7.2 Chain Complexes

7.4 Definition. Let $\mathbf{I}$ be a category with $\operatorname{obj}\mathbf{I} = \mathbb{Z}$ and no morphisms but the identities. Let $\mathbf{C}$ be an abelian category. Then the category $\mathbf{C}^{\mathbf{I}}$ is called the *category of graded objects* of $\mathbf{C}$. It can be identified with the category such that the objects are collections $\{C_i\}_{i\in\mathbb{Z}}$ of objects in $\mathbf{C}$ and a morphism $\{C_i\} \to \{D_i\}$ is a collection of maps $\{\phi_i : C_i \to D_i\}$. We denote this category as $\operatorname{Gr}(\mathbf{C})$.

7.5 Definition. If $\mathbf{C}$ is an abelian category, a *chain complex* $\mathcal{C}$ in $\mathbf{C}$ is a collection $\{C_i\}_{i\in\mathbb{Z}}$ of objects in $\mathbf{C}$ and maps (called *differentials*) $\{d_i : C_i \to C_{i-1}\}_{i\in\mathbb{Z}}$ such that $d_{n+1}d_n = 0$ for all $n \in \mathbb{Z}$. A *chain map* between the chain complexes $\mathcal{C}$ and $\mathcal{C}'$ is a collection of maps $\phi_i : C_i \to C'_i$ such that the following diagram commutes for all $n \in \mathbb{Z}$

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{d_{n+1}} & C_n \\ {\scriptstyle \phi_{n+1}}\downarrow & & \downarrow{\scriptstyle \phi_n} \\ C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n \end{array}$$

[1]**FiXme**: *EXAMPLE: RING IS NOT ABELIAN*
[2]**FiXme**: *If C is abelian, then CI is abelian*
[3]**FiXme**: *EXAMPLES? MAYBE MACKEY FUNCTORS. IE THE CATEGORY OF MACKEY FUNCTORS OVER A GROUP G IS ABELIAN*
[4]**FiXme**: *Define the zero functor*

7.6 *Proposition.* *Chain complexes and chain maps in* $\mathbf{C}$ *form an abelian category which we denote* $\mathrm{Ch}(\mathbf{C})$.

Note that if $F\colon \mathbf{C} \to \mathbf{D}$ is a functor, we can immediately define a functor $\mathrm{Gr}(\mathbf{C}) \to \mathrm{Gr}(\mathbf{D})$, and if F is also additive, we can even define a functor $\mathrm{Ch}(\mathbf{C}) \to \mathrm{Ch}(\mathbf{D})$, which we also denote by F. Also note that we have a forgetful functor $\mathrm{Ch}(\mathbf{C}) \to \mathrm{Gr}(\mathbf{C})$ and a functor $\mathrm{Gr}(\mathbf{C}) \to \mathrm{Ch}(\mathbf{C})$ that sends a graded object to a complex with all differentials to be zero.

We usually have chain complexes in which $C_i = 0$ for $i < 0$, they are called *positive chain complexes*. We could also have a complex for which $C_i = 0$ if $i > 0$, but if this is the case, we call $C^i = C_{-i}$ and $d_{-i}\colon C_{-i} \to C_{-i-1}$ becomes $d^i\colon C^i \to C^{i+1}$. We say then that $\{C^i\}$ is a *cochain complex*. The category of positive chain complexes is denoted $\mathrm{Ch}_{\geq 0}(\mathbf{C})$ and the category of cochain complexes is denoted $\mathrm{Ch}^{\geq 0}(\mathbf{C})$.

[5]

7.3 Exact Sequences

7.7 Definition. Let $f\colon A \to B$ a morphism in the abelian category $\mathbf{C}$. The image of f, denoted $\operatorname{im} f$ is the subobject of B, $\ker(\operatorname{coker} f)$. A sequence of maps in $\mathbf{C}$

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{7.1}$$

is called *exact (at* B*)* if $\ker g = \operatorname{im} f$. A sequence of maps is *exact* if it is exact at every term. A *short exact sequence* in $\mathbf{C}$ is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{7.2}$$

A *map of short exact sequences* is composed of three maps ϕ, χ, ψ in $\mathbf{C}$ such that the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{g_1} & B_1 & \xrightarrow{f_1} & C_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{g_2} & B_2 & \xrightarrow{f_2} & C_2 & \longrightarrow & 0
\end{array}$$

Clearly the short exact sequences in $\mathbf{C}$ form a category which we will denote as $\mathrm{Sh}(\mathbf{C})$. We similarly define a category $\mathrm{LEHS}(\mathbf{C})$ of long exact homology sequences, which is the same as the category of exact positive chain complexes. Also we define the category $\mathrm{LECS}(\mathbf{C})$ of long exact cohomology sequences, which are the exact cochain complexes.

[5] **FiXme**: *Define chain homotopy equivalence*

7.4 Homology

7.8 Definition. If $\mathcal{C}$ is a chain complex, we define the subobjects of C_n: $Z_n(\mathcal{C}) = \ker d_n$ (n-*cycles*), $B_n(\mathcal{C}) = \operatorname{im} d_{n+1}$ (n-*boundaries*), and the subquotient $H_n(\mathcal{C}) = Z_n(\mathcal{C})/B_n(\mathcal{C})$ (n-*homology*). For a cochain complex $\mathcal{C}$, we have the subobjects of C^n: $Z^n(\mathcal{C}) = \ker d^n$ (n-*cocycles*), $B^n(\mathcal{C}) = \operatorname{im} d^{n-1}$ (n-*coboundaries*), and the subquotient [6] $H^n(\mathcal{C}) = Z^n(\mathcal{C})/B^n(\mathcal{C})$ (n-*cohomology*)

Observe that $\mathcal{C}$ is exact at C_n precisely when $H_n(\mathcal{C}) = 0$. Also we have that Z_n, B_n, H_n define functors $\mathrm{Ch}(\mathbf{C}) \to \mathbf{C}$, and all together form functors $Z_*, B_*, H_*\colon \mathrm{Ch}(\mathbf{C}) \to \mathrm{Gr}(\mathbf{C})$.

7.9 *Proposition.* ([Wei94, page 7]) *A sequence*

$$0 \to \mathcal{C}' \to \mathcal{C} \to \mathcal{C}'' \to 0 \tag{7.3}$$

is exact in $\mathrm{Ch}(\mathbf{C})$ *if and only if each sequence*

$$0 \to C'_n \to C_n \to C''_n \to 0 \tag{7.4}$$

is exact in $\mathbf{C}$.

[7]

7.5 Split Chain Complexes

7.10 Definition. We say that the chain complex $\mathcal{C}$ is split if there is a map $s\colon \mathcal{C} \to \mathcal{C}$ of degree 1 such that $dsd = d$.

7.11 *Proposition.* ([Web87, page 363]) *Let* $\mathcal{C}$ *be a chain complex. The following conditions are equivalent:*

1. $\mathcal{C}$ *is split,*
2. *For every* n, *both* $d_n\colon C_n \to \operatorname{im} d_n$ *is a split epimorphism and the inclusion* $\operatorname{im} d_n \to C_{n-1}$ *is a split monomorphism.*

[7] **FiXme**: *put relationship with chain homotopy equivalence*

3. *For every* n *we can write*

$$C_n \cong \operatorname{im} d_{n+1} \oplus H_n(\mathcal{C}) \oplus \operatorname{im} d_n$$

so that d_n *becomes the map from* $\operatorname{im} d_{n+1} \oplus H_n(\mathcal{C}) \oplus \operatorname{im} d_n$ *to* $\operatorname{im} d_n \oplus H_{n-1}(\mathcal{C}) \oplus \operatorname{im} d_{n-1}$ *sending* $(a, \overline{b}, c)$ *to* $(c, 0, 0)$.

Proof. To prove 2 implies 3, let s_n be a splitting of d_n and s'_n be a splitting of the inclusion. We have, using s_n, that $C_n \cong \ker d_n \oplus \operatorname{im} d_n$. The map s'_n that gives the splitting of the inclusion can then be restricted to give a splitting of the inclusion $\operatorname{im} d_{n+1} \to \ker d_n$. Then, we have the isomorphism

$$\begin{array}{ccc} C_n & \longrightarrow & \operatorname{im} d_{n+1} \oplus H_n(\mathcal{C}) \oplus \operatorname{im} d_n \\ x & \longmapsto & (s'_n x - s'_n s_n d_n x, \overline{x - s_n d_n x}, d_n x) \end{array}$$

and that it has inverse

$$\begin{array}{ccc} \operatorname{im} d_{n+1} \oplus H_n(\mathcal{C}) \oplus \operatorname{im} d_n & \longrightarrow & C_n \\ (a, \overline{b}, c) & \longmapsto & a + b - s'_n(b) + s_n(c) \end{array}$$

We can then check that d_n has the desired rule of correspondence.

To prove 3 implies 1, define $s_n(x, y, z) = (z, 0, 0)$. □

7.12 *Corollary.* *Let* $\mathcal{C}$ *be a chain complex. The following conditions are equivalent:*

1. $\mathcal{C}$ *is split and exact.*
2. $\mathcal{C}$ *is chain homotopy equivalent to the zero complex, that is,* $\mathcal{C} \simeq 0$.

Proof. Suppose that $\mathcal{C}$ is split and exact, then by 3 from the previous proposition, we have that $C_n \cong \operatorname{im} d_{n+1} \oplus \operatorname{im} d_n$, and the expression for the boundary is $(a, c) \mapsto (c, 0)$. Define $t_n\colon \operatorname{im} d_{n+1} \oplus \operatorname{im} d_n \to \operatorname{im} d_{n+2} \oplus \operatorname{im} d_{n+1}$ by $t_n(a, c) = (0, a)$. It is immediate to check that $dt + td = 1$. Now, if $\mathcal{C} \simeq 0$, let $t\colon C_n \to C_{n+1}$ such that $dt + td = 1$. Composing on the right with d, we get $dtd = d$, and so $\mathcal{C}$ is split. To check $\mathcal{C}$ is exact, take $x \in C_n$ such that $d_n x = 0$. Hence, applying $t_{n-1} d_n + d_{n+1} t_{n+1} = 1_{C_n}$ to it, we get $d_{n+1} t_{n+1} x = x$, that is, x is a boundary. Hence $H_n(\mathcal{C}) = 0$. □

7.13 Definition. If the chain complex $\mathcal{C}$ satisfies one of the conditions of the previous corollary, we say it is *contractible*.

7.6 Exact Functors

7.14 Definition. Let $F\colon \mathbf{C} \to \mathbf{D}$ be an additive covariant functor between abelian categories. Then F is called *left exact* if for any exact sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, we have that the following sequence is exact:

$$0 \to FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC. \tag{7.5}$$

Now, F is *right exact* if for any exact sequence of the form $A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the following sequence is exact:

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \to 0. \tag{7.6}$$

The functor is *exact* if it is both left and right exact.
A contravariant additive functor $F\colon \mathbf{C} \to \mathbf{D}$ is left exact (right exact, exact) if the corresponding covariant functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ is left exact (right exact, exact). That is, the contravariant F is left exact if for any exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ we have that $0 \to FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$ is exact, and it is right exact if for any exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ we have that $FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \to 0$ is exact.

Let $\mathbf{C}$ be an abelian category and M be an object in $\mathbf{C}$. Then we can consider the hom functor $\hom_{\mathbf{C}}(M, -)$ as taking values in the category $\mathbf{Ab}$. Similarly, we have a contravariant hom functor $\hom_{\mathbf{C}}(-, M)\colon \mathbf{C} \to \mathbf{Ab}$. Both are clearly additive.

7.15 ***Theorem.*** ([Wei94, pages 27–28]) *Let $\mathbf{C}$ be an abelian category. Then for any object M in $\mathbf{C}$ we have that both $\hom_{\mathbf{C}}(M, -), \hom_{\mathbf{C}}(-, M)\colon \mathbf{C} \to \mathbf{Ab}$ are left exact.*

[8]

7.16 ***Theorem.*** ([Wei94, page 28]) *Let $\mathbf{C}$ be an abelian category. A sequence in $\mathbf{C}$*

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{7.7}$$

is exact if the sequence

$$\hom(M, A) \xrightarrow{f_*} \hom(M, B) \xrightarrow{g_*} \hom(M, C) \tag{7.8}$$

is exact for every $M \in \operatorname{obj} \mathbf{C}$.

[8] **FiXme**: *GIVE PROOF*

7.17 *Theorem.* ([Wei94, page 51]) *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be abelian categories, and* $R\colon \mathbf{C} \to \mathbf{D}$, $L\colon \mathbf{D} \to \mathbf{C}$ *be additive functors such that* $L \dashv R$. *Then*

1. L *is right exact*
2. R *is left exact.*

7.18 Definition. A non-empty small category $\mathbf{I}$ is called *filtered* if:

1. For every $i, j \in \operatorname{obj} \mathbf{I}$ there is $k \in \operatorname{obj} \mathbf{I}$ and morphisms $i \to k$, $i \to j$.
2. For every $i, j \in \operatorname{obj} \mathbf{I}$ and every two morphisms $u, v \in \hom_{\mathbf{I}}(i, j)$, there is $k \in \operatorname{obj} \mathbf{I}$ and a morphism $w \in \hom_{\mathbf{I}}(j, k)$ such that $wu = wv$.

7.19 *Theorem.* ([Wei94, page 58]) *If* $\mathbf{I}$ *is filtered, then both functors* $\operatorname{colim}, \lim\colon (R\text{-}\mathbf{mod})^{\mathbf{I}} \to R\text{-}\mathbf{mod}$ *are exact*

[9]

[9] **FiXme**: *NOW We could later include that Q IS FLAT, WHICH USES COR 2.6.17 IN WEIBEL.*

8

Appendix: Calculations

8.1 The Grothendieck Group

A semigroup is a set S together with a commutative and associative binary operation on S. A morphism $f\colon S_1 \to S_2$ of abelian semigroups is a map $f\colon S_1 \to S_2$ that preserves the operation. We will use **Abs** to denote the category of abelian semigroups.

8.1 Definition. ([HS97, page 72]) Let S be an abelian semigroup. Then $S \times S$ has an induced structure of abelian semigroup. Define in $S \times S$ the relation $(a, b) \sim (c, d)$ if and only if there is $u \in S$ such that $a+d+u = b+c+u$. This is an equivalence relation, and $(S\times S)/\!\sim$ is an abelian group, since $[a, b]+[b, a] = [a+b, a+b] = [0, 0] = 0$. This is the *Grothendieck group* of S, denoted $\mathrm{Gr}(S)$. Note that we have a homomorphism $i\colon S \to \mathrm{Gr}(S)$. We observe the following universal property of i: If A is an abelian group and $f\colon S \to A$ is a homomorphism, then there is $\overline{f}\colon \mathrm{Gr}(S) \to A$ making commute the following diagram

$$\begin{array}{ccc} S & \xrightarrow{\;i\;} & \mathrm{Gr}(S) \\ {\scriptstyle f}\downarrow & \swarrow_{\overline{f}} & \\ A & & \end{array} \tag{8.1}$$

This universal property lets us verify that if $E\colon \mathbf{Ab} \to \mathbf{Abs}$ is the forgetful functor, we have that $\mathrm{Gr} \dashv E$

Examples **8.1**

1. If S is the semigroup of all G-sets under disjoint union, then Gr(S) is the additive group of what is called the *Burnside ring* of G and is denoted $\Omega(G)$,
2. If S is the semigroup of finitely generated kG-modules under direct sum, then Gr(S) is the additive group of what is called the *Green ring* of G over k.

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